Geometric Bookkeeping for Feynman Integrals

Systematic Evaluation from Hodge Theory

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Milestones in Amplitudes for Gravitational Waves and Cosmology
ETH-ITS Zurich
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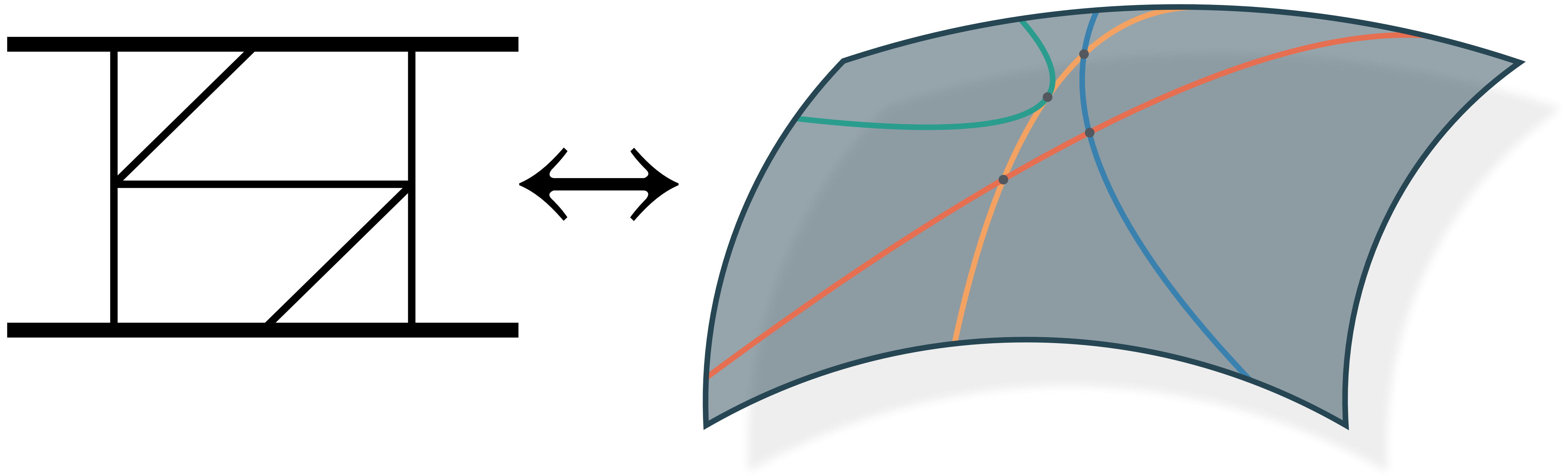


Based on 2412.12057, 2506.09124, 2507.23594, 2511.15381
with ε-collaboration:
Iris Bree, Federico Gasparotto, Antonela Matijašić, Pouria Mazloumi, Dmytro Melnichenko,
Toni Teschke, Xing Wang, Stefan Weinzierl, Konglong Wu, Xiaofeng Xu

as well as Hjalte Frellesvig, Roger Morales, Matthias Wilhelm

Roadmap of this talk

Introduction to Feynman integrals

Differential equations and ε-factorized form

Feynman integrals with non-trivial geometries

What is new: Algorithmic procedure for ε-factorized differential equations

Twisted cohomology and Baikov representation

Filtrations from Hodge theory

Core of the algorithm

Examples

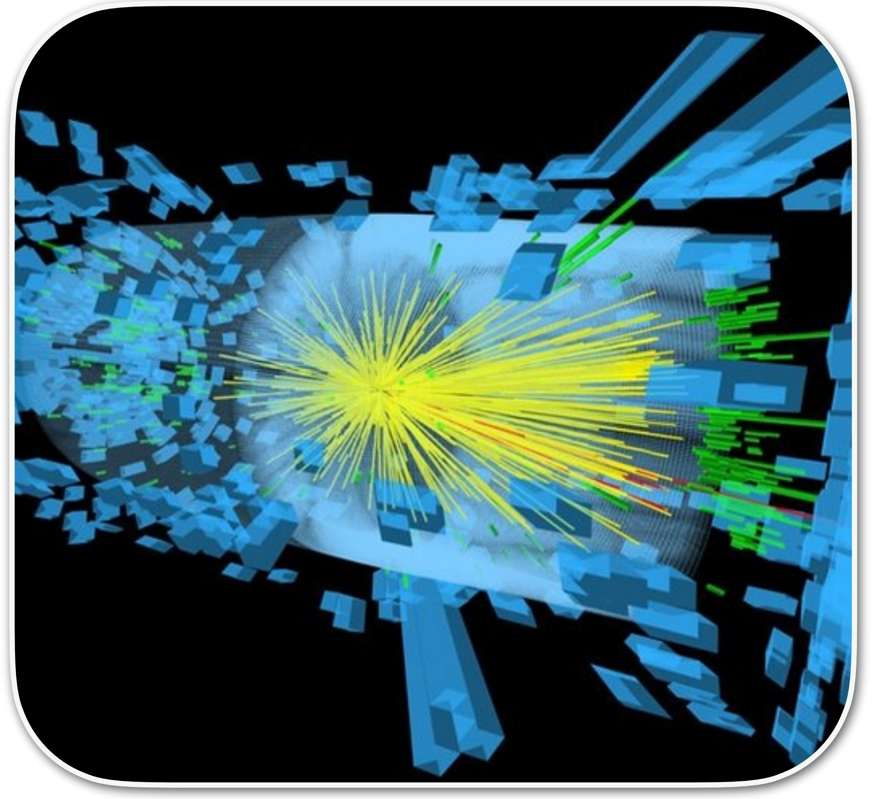
Three-loop Banana integral with four scales

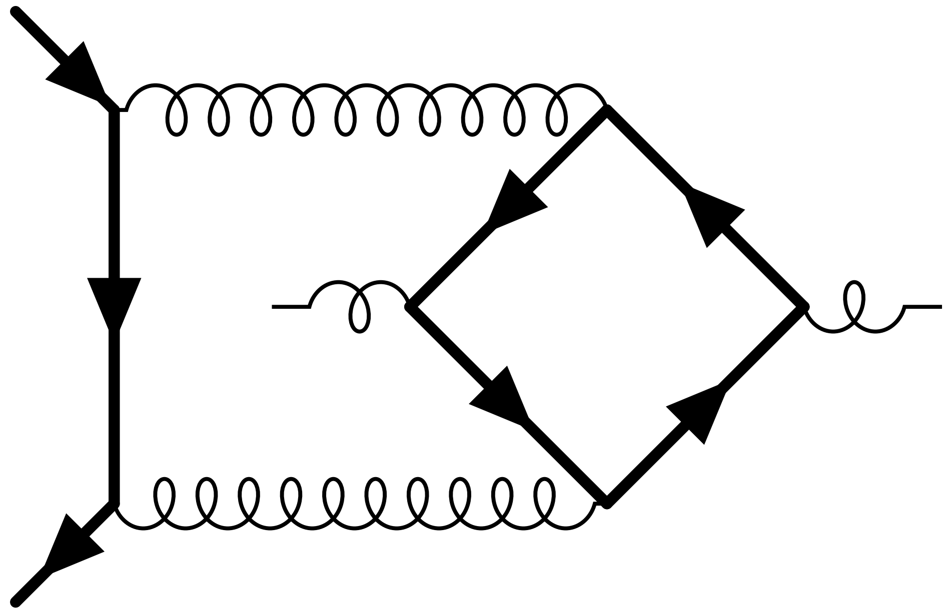
Introduction

Feynman Integrals

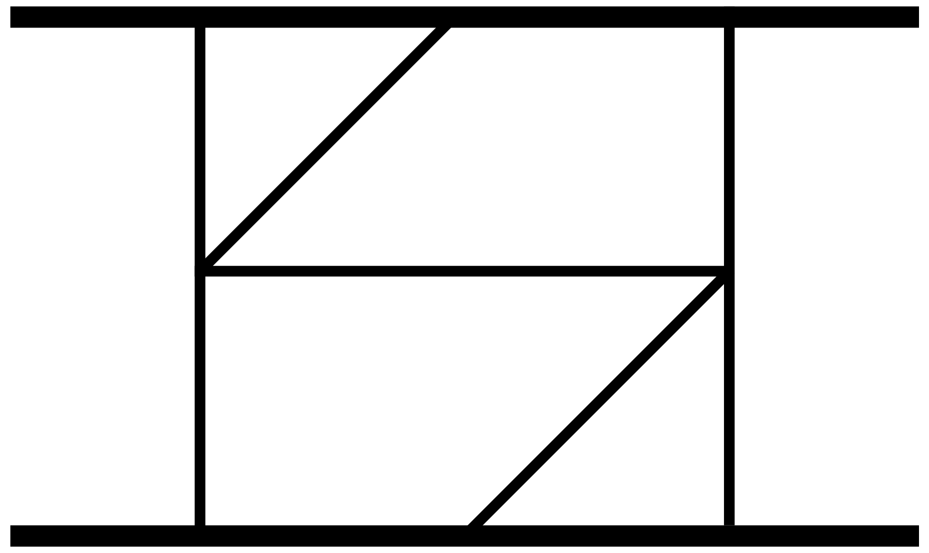
Building blocks of scattering amplitudes

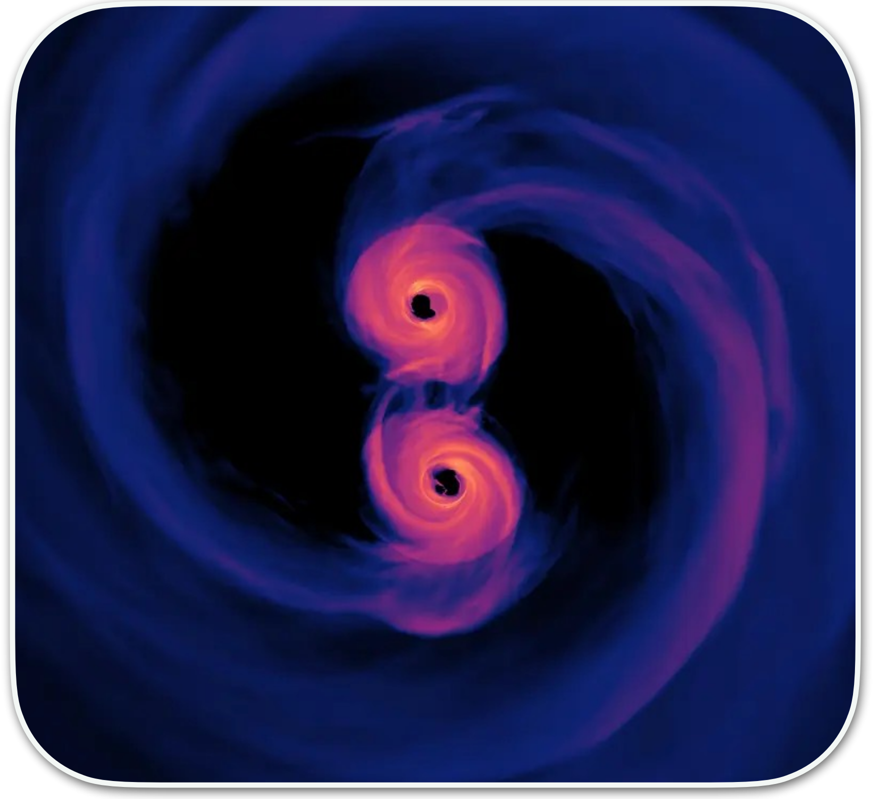
Particle Physics





Gravity





Scattering of…

Particles

Black holes

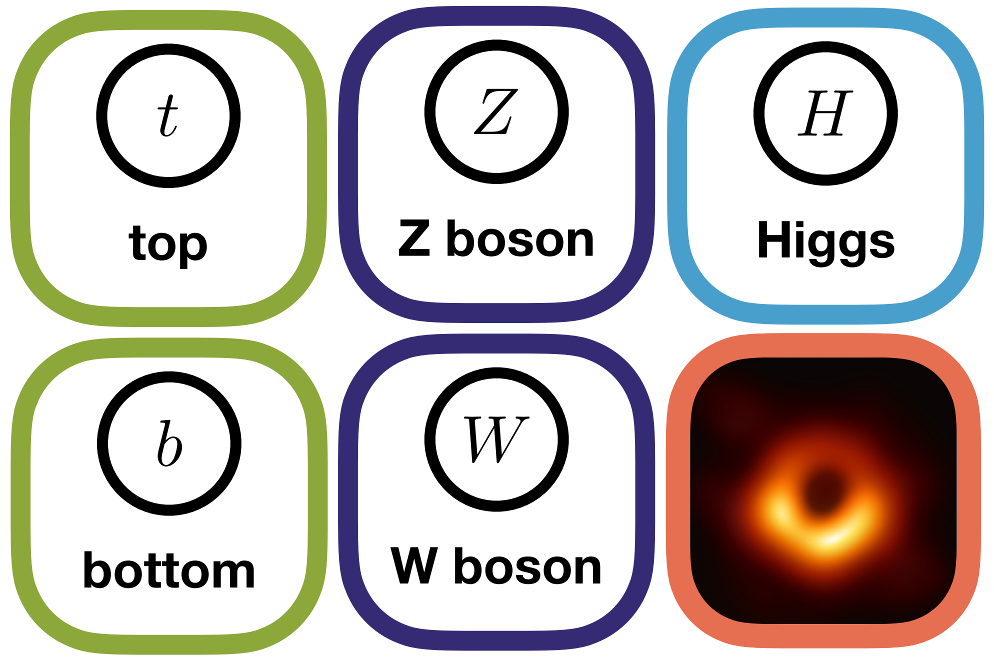
Theory independent building blocks capturing loop-level analytic complexity

Where do the challenges lie?

Feynman integrals scale in complexity with number of…

Kinematic scales & Loops

$$\simeq$$



/ high multiplicity @ high-precision

Feynman Integrals 101

Dimensional regularization
$d = d_0 - \boxed{2\varepsilon}$

$$I_{\nu_1 \ldots \nu_m} = \int \prod_i \frac{\mathrm{d}^d l_i}{i\pi^{d/2}} \frac{1}{\prod_j D_j^{\nu_j}}$$

Propagators $\nu_j \in \mathbb{Z}$
Numerators: $\nu_j < 0$

What we want:

Laurent series of $I_{\nu_1 \ldots \nu_m}$ in ε

Feynman Integrals: Some properties

Two important features:

① Integration-by-parts relations $$\int \left(\prod_i \mathrm{d}^D l_i\right) q^\mu \frac{\partial}{\partial l_j^\mu}\left(\frac{1}{\prod_j D_j^{\nu_j}}\right) = 0$$

Total derivative

→Generate linear relations between Feynman integrals

→Can find a minimal basis of Feynman integrals: **Master Integrals**

② Derivatives of Feynman Integrals are again Feynman Integrals

w.r.t. external kinematics

Differential equations for Feynman Integrals

Main tool to evaluate Feynman Integrals

Basis of Master Integrals $I = \{I_1, \dots, I_n\}$
Kinematic variables $x = \{x_1, \dots, x_n\}$

Matrix of differential 1-forms

$$\mathrm{d}I = A(x,\varepsilon)I$$

Find “good” basis $J = UI$ such that ε factorizes

$$\mathrm{d}J = \varepsilon\tilde{A}(x)J$$

Solution given by path-ordered exponential

$$J = \mathbb{P}\exp\left(\varepsilon\int\tilde{A}\right)J_0$$

Let $\mathcal{C}(t)$ be an integration contour
with $t \in [0,1]$ $\mathcal{C}(0) = x_0$ $\mathcal{C}(1) = x$

$$J = \varepsilon^0 J_0 + \varepsilon^1\int_0^1 \mathrm{d}t\tilde{A}(t)J_0 + \varepsilon^2\int_0^1 \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\tilde{A}(t_1)\tilde{A}(t_2)J_0 + \mathcal{O}(\varepsilon^3)$$

Why ε-factorized differential equations?

Provide direct access to…

…solution at any order in ε

…required function space ($\tilde{A}$ entries)

…singularity structure

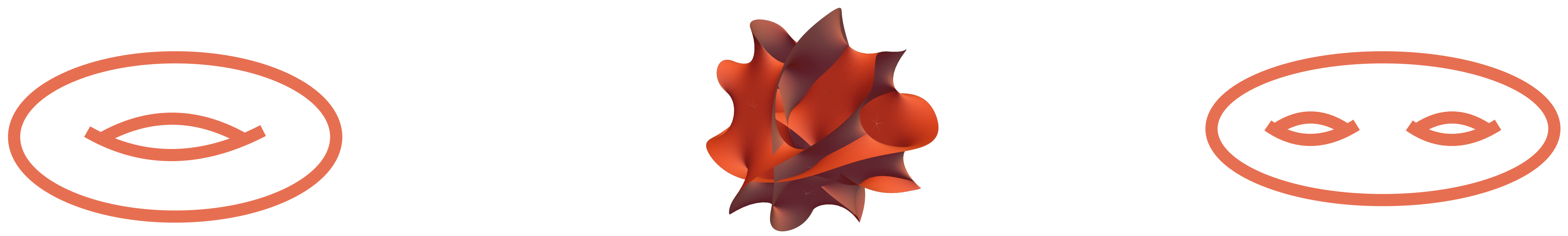
In summary:
Make important properties of Feynman integrals manifest

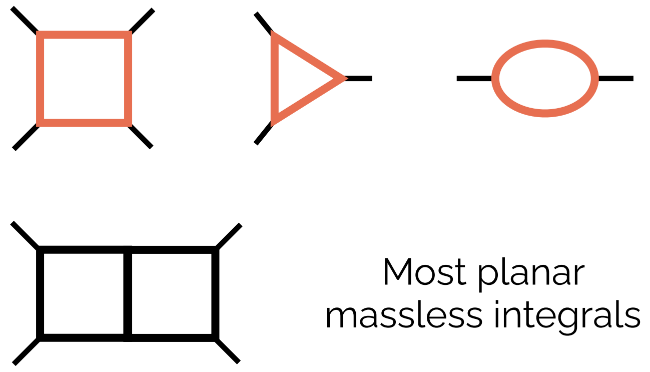
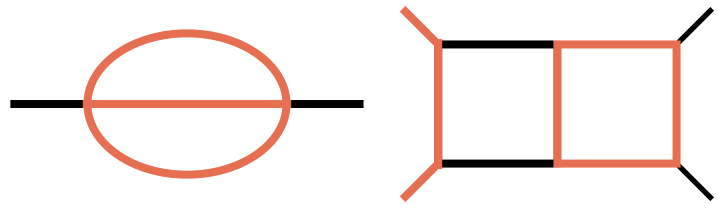
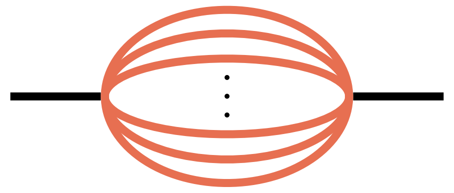
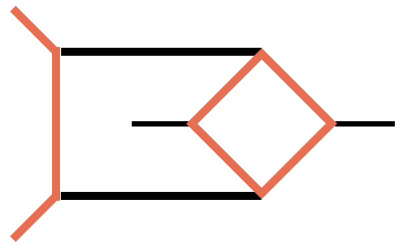
General methods for deriving ε-factorized differential equations are key to systematic understanding

A Zoo of Geometries

Integrals associated to geometries

Determines functions appearing in $\mathrm{d}J = \varepsilon \tilde{A}(x) J$



Geometry	$\mathbb{P}^1$	Elliptic curves	Calabi—Yaus	Higher-genus Curves	...
Functions	(poly)logarithms $\log$, Li_n, MPL	Elliptic functions K, E, Π, eMPL modular forms	? Expansions, in some cases modular forms	Higher-genus functions heMPL Siegel modular forms	more interesting objects
Examples					

At two-loop full classification done

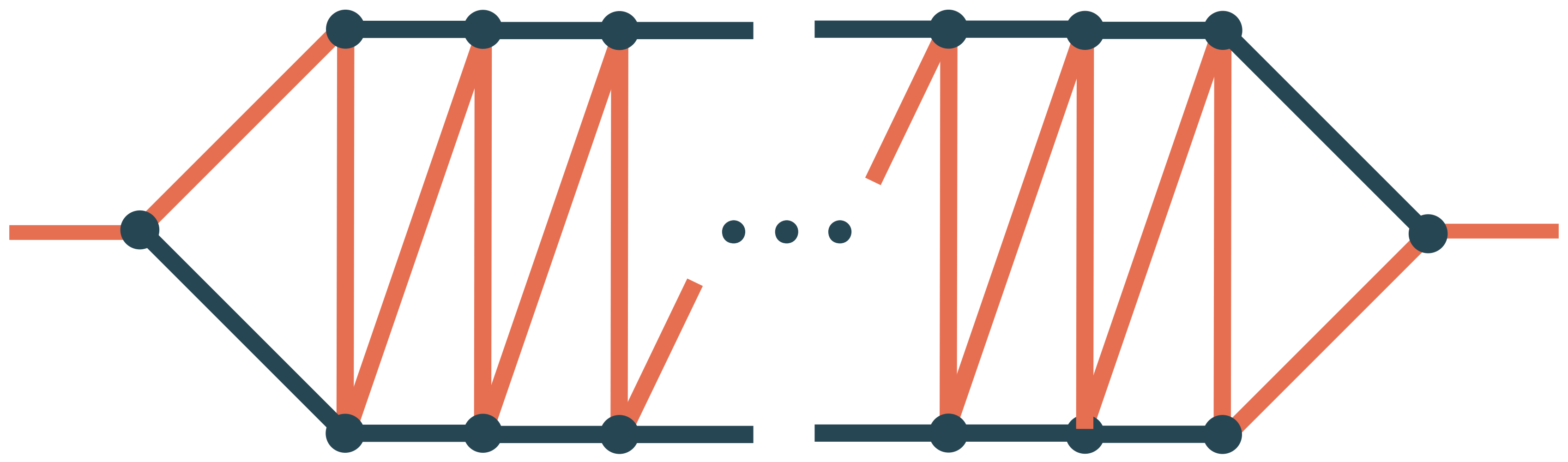
[Bargiela, Frellesvig, Marzucca, Morales, Seefeld, Wilhelm, Yang, 2512.13794]

Calabi—Yaus in Feynman Integrals are not at all special

QED

e^- : massive

γ : massless

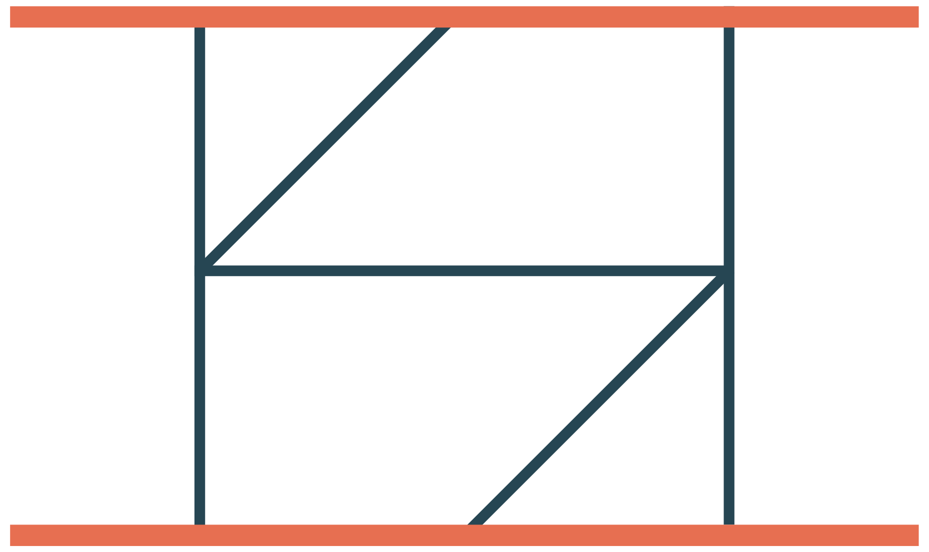


$2\ell \rightarrow$ Calabi—Yau $(2\ell\text{-}1)$ fold

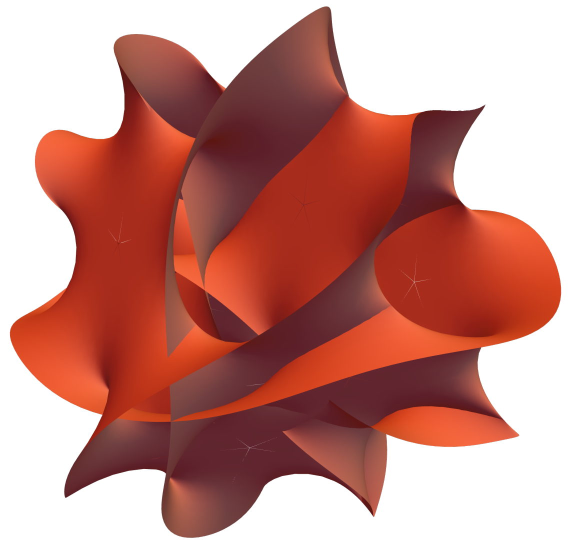
Calabi—Yaus in Gravity

Calabi—Yau integrals play a central role in post-Minkowskian expansion

e.g. at 5PM, 2SF (see talks by Jan and Enrico)



$\simeq$



Calabi—Yau 3-fold

First identified

[Frellesvig, Morales, Wilhelm, '23]

ε-factorized DEQ derived in

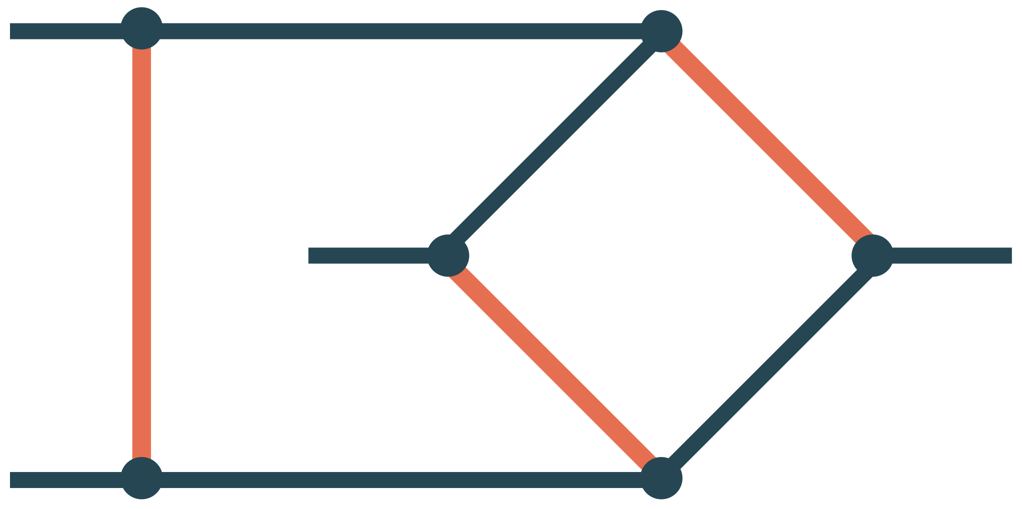
[Frellesvig, Morales, SP, Weinzierl, Wilhelm, '24]

More examples

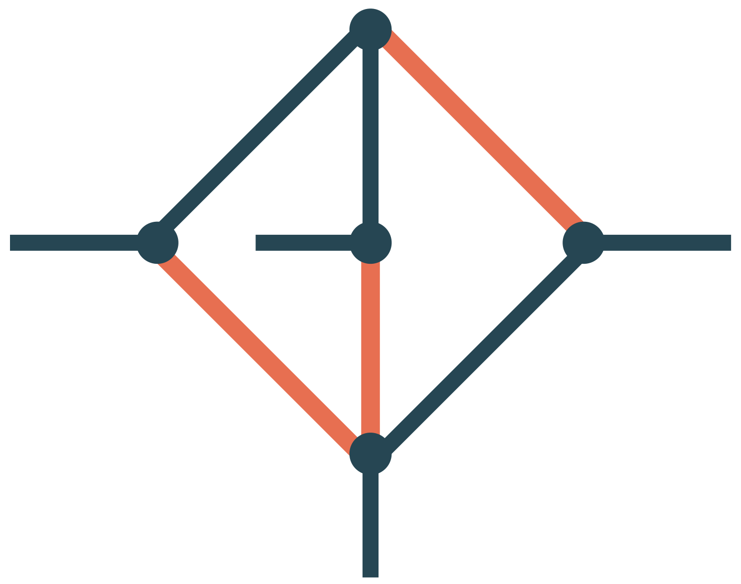
Møller ($e^-e^- \to e^-e^-$) scattering at two-loops

e^-: massless

Z : massive



Genus 2
hyperelliptic curve



K3 surface
(Calabi—Yau 2-fold)

Already at two-loop:
non-trivial geometries become prominent

Need new tools to simplify analytic evaluation of such integrals

Non-trivial geometries and ε-factorized DEQs

Where we are vs. where we want to be

With current geometry specific approaches…

- Requires case-by-case analysis
- Required special functions can be poorly understood
  (also in mathematics literature)
- Requires specialized knowledge on the side of users

We would like a method that…

- … does not require advanced knowledge of special functions
- … is truly algorithmic (no user input required)
- … naturally extends to new geometries and many kinematic scales

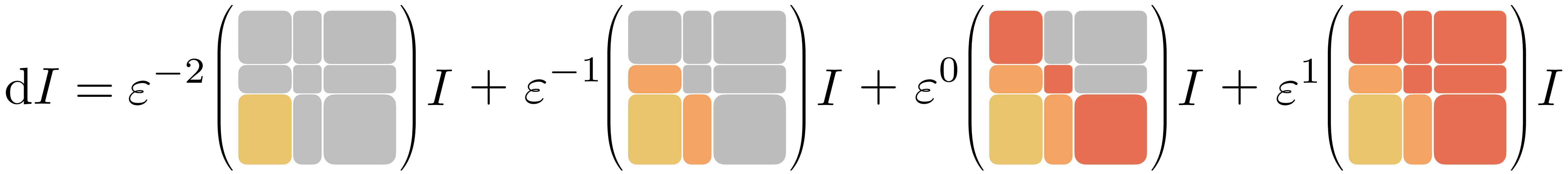
The Algorithm

Two step process to derive ε-factorized differential equations

on maximal cut

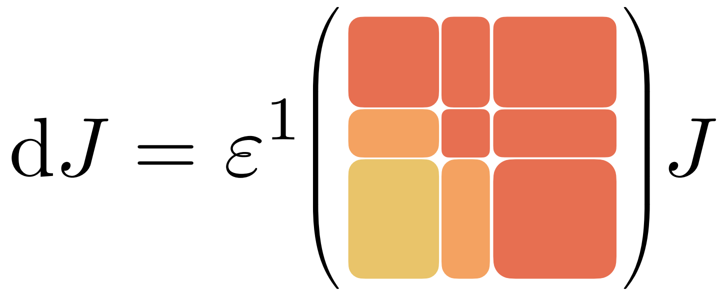
[ε-collaboration, 2506.09124, 2511.15381]

① Choose Master Integrals, such that DEQ is Laurent polynomial in ε



② Rotate away non-ε-factorizing pieces, leaving only ε^1 term

$J = R^{-1}I$ such that



Both steps algorithmic!

1

Choosing Masters with Laurent polynomial DEQ

Framework: Twisted Cohomology

Standard de Rham cohomology

$$\int \varphi$$

rational differential form

Integration by Parts: $\varphi \simeq \varphi + \mathrm{d}\xi$

Not most natural to capture Feynman integrals in dimensional regularization.
Instead…

Twisted de Rham cohomology

$$\int u\varphi$$

rational differential form

Fixed multivalued function
Twist

Integration by Parts: $\varphi \simeq \varphi + \nabla_u \xi$ with $\nabla_u = d + \mathrm{d}\log(u)\wedge$

Twisted Cohomology and Feynman Integrals

Baikov representation of Feynman Integrals

Propagators $\equiv$ Integration Variables

$q_j = a_j/2 + b_j\varepsilon$

$$I_{\nu_1\ldots\nu_n} = C_B \int \underbrace{\left(\prod_j P_j^{q_j}\right)}_{\text{twist } u \text{ regulates singularities at } P_j = 0} \prod_i \frac{\mathrm{d}z_i}{z_i^{\nu_i}}$$

$z_i \equiv D_i$

Defines characteristic polynomial(s)

Baikov polynomials P_j

Encode all geometric information of integral

Together with exponents a_j, b_j

Distinguish Baikov polynomials

$$a_j \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix} \to P_j \begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$$

In the following, we work entirely in Baikov space

Forget about Feynman integrals for the moment

Consider generic integrands of the form

$$\int u\Phi_{\mu_1\ldots\mu_k}[Q]$$

Baikov twist $u = \prod_j P_j^{q_j}$

with

Polynomial in z_i

$$\Phi_{\mu_1\ldots\mu_k}[Q] = C(\varepsilon)\frac{Q(z_i)}{P_1^{\mu_1}\ldots P_k^{\mu_k}}\mathrm{d}z_1\wedge\ldots\wedge\mathrm{d}z_n$$

Can always translate back to Feynman integrals whenever needed

Baikov Representation and Hodge Theory

Baikov representation suggests understanding differential forms…

…in $\mathbb{CP}^n$ (after suitably projectivizing)
…with set of characteristic polynomials P_j
…with singularities only at $P_j = 0$

Without twist this situation is already understood:

Mixed Hodge Structure

[Deligne, 1970]

$\simeq$

(for our purposes)

Classify differential forms with support on hypersurfaces and their intersections

Equivalent twisted version not clear…
However, ideas carry over!

Geometric Integrands in Baikov Representation

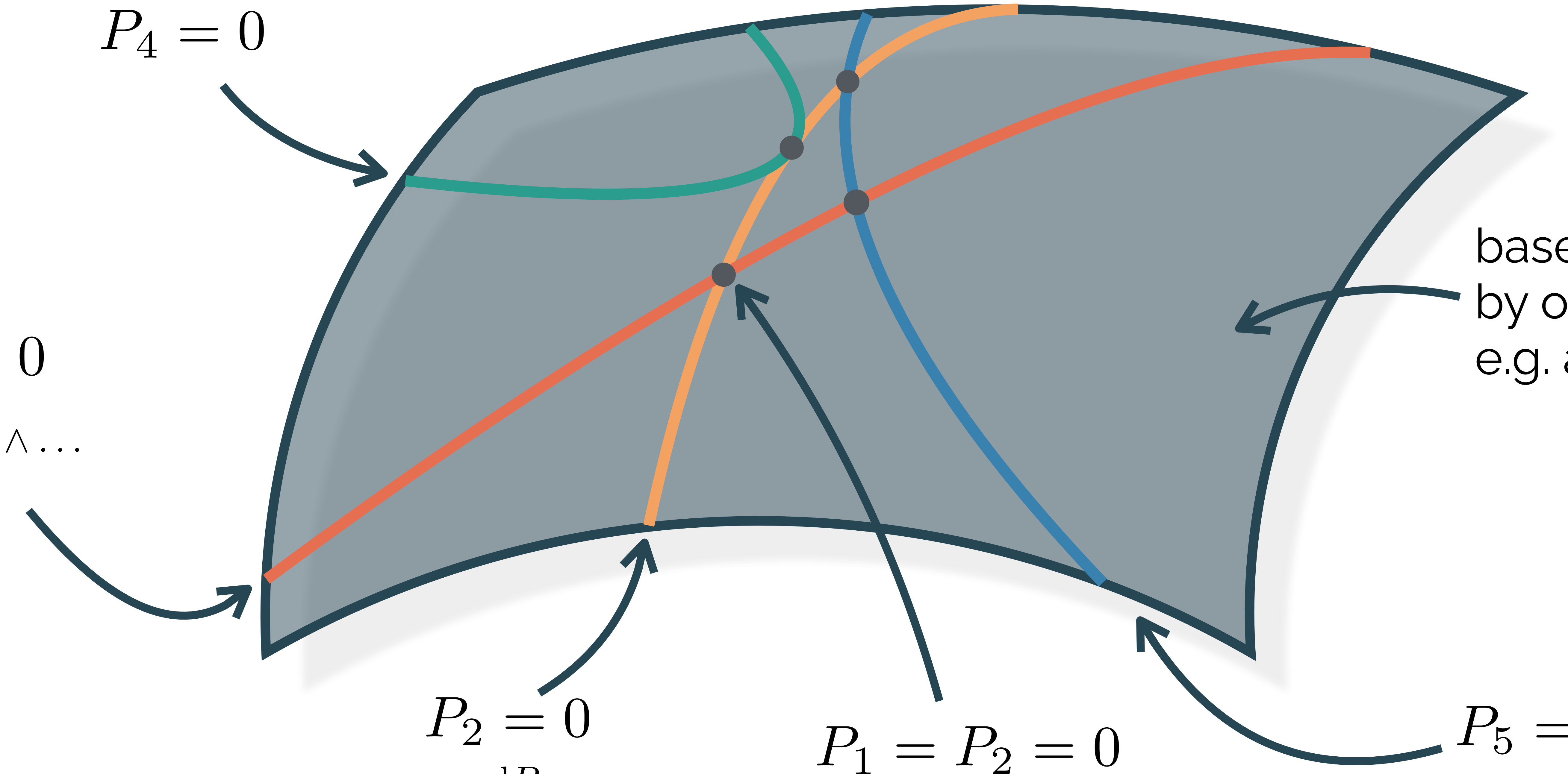
Base space $\mathbb{CP}^n$, hypersurfaces of interest $P_j = 0$ (even)

Build integrands with (locally) logarithmic singularities

Taking residues localizes on hypersurfaces

→ construct forms with support on hypersurfaces

only possible for
even Baikov polynomials
$u \propto P_j^{b_j \varepsilon}$



$P_4 = 0$

$P_1 = 0$

$\Phi = \frac{\mathrm{d}P_1}{P_1} \wedge \ldots$

(locally)

$P_2 = 0$

$\Phi = \frac{\mathrm{d}P_2}{P_2} \wedge \ldots$

(locally)

$P_1 = P_2 = 0$

$\Phi = \frac{\mathrm{d}P_1}{P_1} \wedge \frac{\mathrm{d}P_2}{P_2} \wedge \ldots$

(locally)

$P_5 = 0$

base geometry given
by odd Baikov polynomials
e.g. a Calabi—Yau

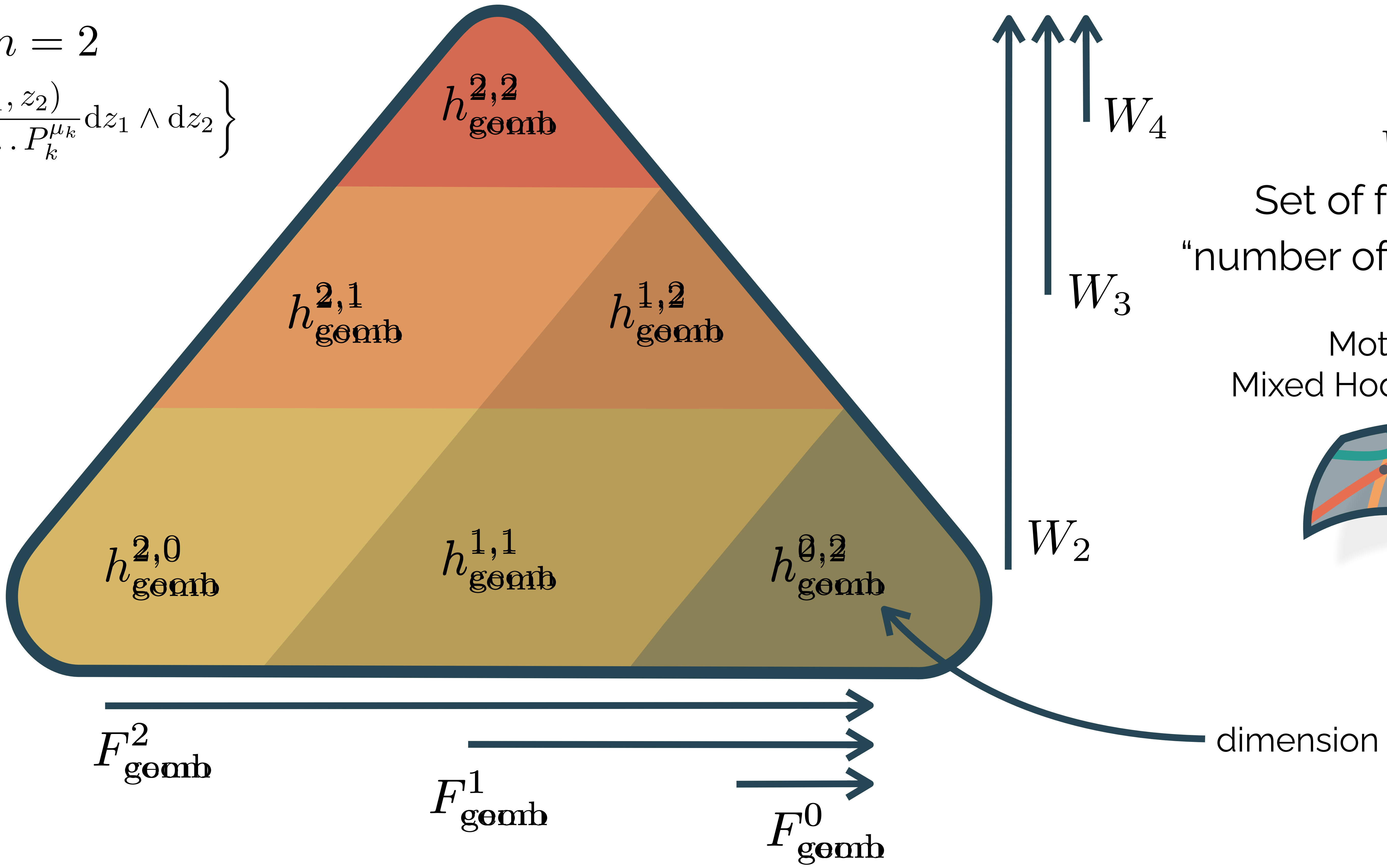
$$y^2 - \prod_{a_j \text{ odd}} P_j = 0$$

Filtrations

Tool to classify differential forms depending on certain properties

Example: $n = 2$

$$\Omega^2 = \left\{ \frac{Q(z_1, z_2)}{P_1^{\mu_1} \ldots P_k^{\mu_k}} \mathrm{d}z_1 \wedge \mathrm{d}z_2 \right\}$$



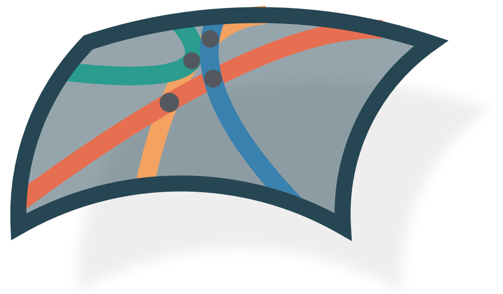
$W_\bullet$

Set of forms with

"number of residues $\geq$ x"

Motivation:

Mixed Hodge Structure



$F^\bullet_{\text{geom}}$ $F^\bullet_{\text{comb}}$

Set of forms with

"order of deepest pole $\geq$ x"

"number of $P_j \geq$ x"

Motivation:

Griffiths transversality

empirical observation

$\partial F^n_{\text{comb}} \subset F^{n-1}_{\text{comb}}$

ε-Prefactors

Reminder: We consider differential forms

$$\Phi_{\mu_1\ldots\mu_k}[Q] = C(\varepsilon)\,\frac{Q(z_i)}{P_1^{\mu_1}\ldots P_k^{\mu_k}}\,\mathrm{d}z_1\wedge\ldots\wedge\mathrm{d}z_n$$

need a prescription for ε prefactors

Three components, dependent on μ_i

$$C(\varepsilon) = C_{\text{abs}}\times C_{\text{rel}}\times C_{\text{clutch}}$$

C_{abs}	C_{rel}	C_{clutch}
$C_{\text{abs}}\times C_B$ pure transcendental weight zero	$\prod_i \left(\frac{1}{2}(a_i+b_i\varepsilon)\right)_{\mu_i}$	$\varepsilon^{-\sum\mu_i}$

All IBP relations take the form:

$$\Phi_{\mu_1\ldots\mu_i\ldots\mu_k}\left[\partial_{z_j}Q\right] + \varepsilon\sum_i \Phi_{\mu_1\ldots(\mu_i+1)\ldots\mu_k}\left[Q\cdot\left(\partial_{z_j}P_i\right)\right] = 0$$

Q independent of ε

ε-dependence trivialized

Integration-by-Parts

Heavy lifting of the algorithm done via IBP reduction

Integration-by-Parts: Quick reminder

$$\int u\nabla_u\xi = 0 \longrightarrow \text{Linear relations between integrals} \xrightarrow{\text{Row reduction}} \text{Basis of integrals}$$

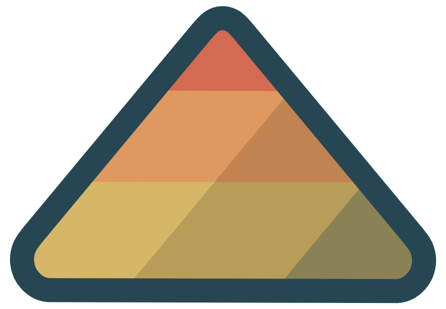
Row reduction requires ordering of integrals

Choose ordering that prefers integrands with “good” properties!

1. Prefer integrands with more consecutive logarithmic singularities
2. Prefer integrands with lower deepest poles
3. Prefer integrands with fewer powers of P_j in denominator

$$\left.\begin{array}{l}1\\2\\3\end{array}\right\}$$

Determined by filtrations



To recap…

1. Derive Baikov representation of Feynman integral with polynomials P_j
2. Build generic differential forms in Baikov space
3. Classify forms with Hodge theoretic filtrations
4. Define ε-prefactors
5. Based on filtrations, define integral ordering
6. Perform IBP reduction with that ordering

Laurent Polynomial Differential Equations

Claim:

Basis of integrands chosen by these criteria satisfies a Laurent polynomial differential equation

$$\mathrm{d}I = \sum_{p=p_{\min}}^{1} \varepsilon^{p} A^{(p)} I$$

Independent of ε

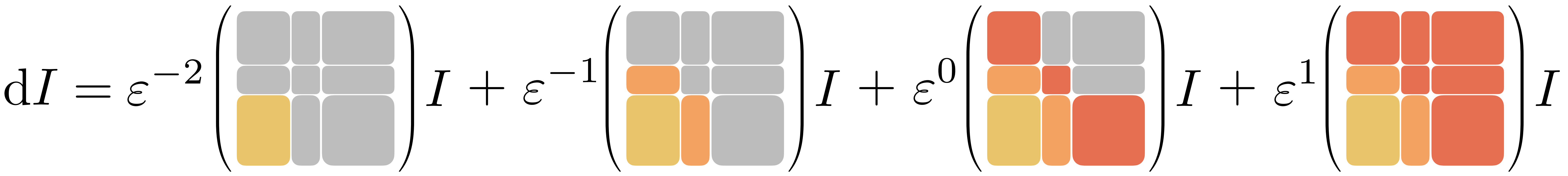
More than that:

The orders in ε satisfy a block structure imposed by $F^{\bullet}_{\mathrm{comb}}$

$$\mathrm{d}I_i = \sum_{p=-(|\mu|_i-|\mu|_j)}^{1} \varepsilon^{p} A^{(p)}_{ij} I_j$$

Total power of P_j

Example: basis with largest $|\mu| = 2$ and smallest $|\mu| = 0$



For completeness

- Need to take care of symmetry relations
- May have to include super sectors

Even so:

All steps **algorithmic** and **purely rational!**

If you care about numerics, can decide to stop here

Laurent polynomial form is great for numeric DEQ solving, e.g. via DiffExp

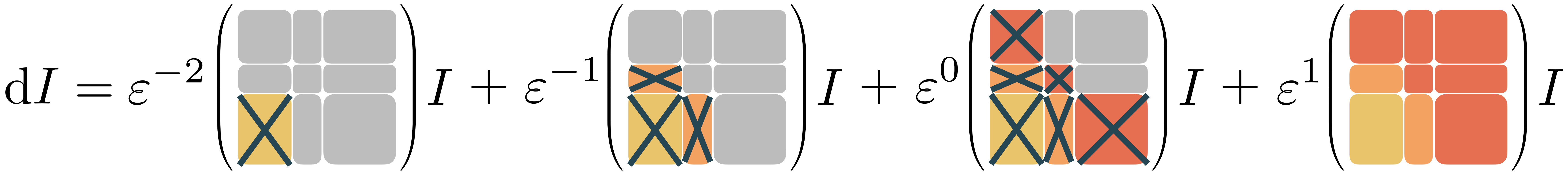
Special functions only get introduced going from
Laurent polynomial to ε-factorized form

2

From Laurent Polynomial to ε-factorized form

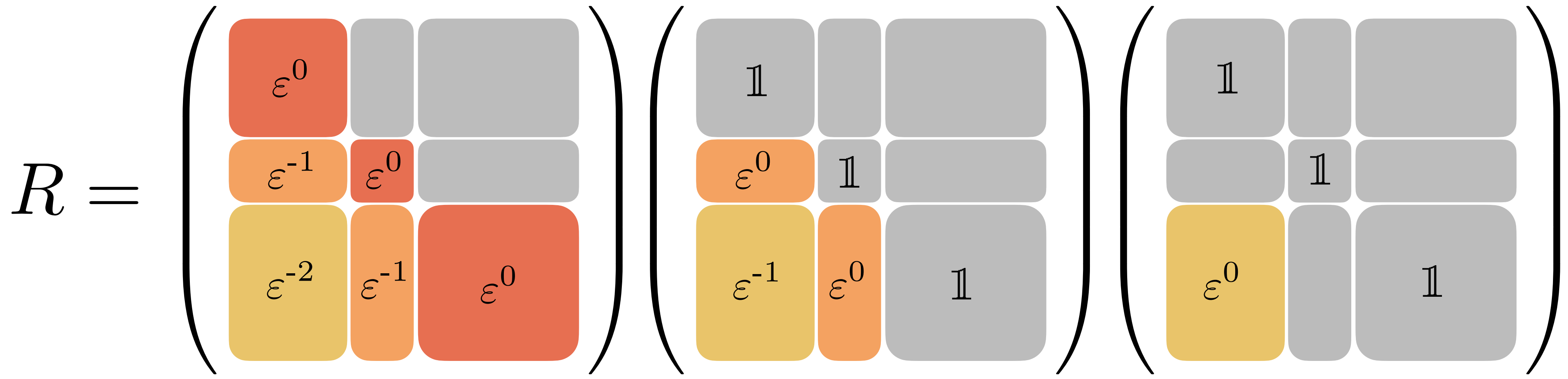
From Laurent polynomial to ε-factorized form

Starting from Laurent polynomial, can systematically eliminate
non-ε-factorizing terms



Define $K = R^{-1} I$

R matrix of generic functions in kinematics, but fixed ε-dependence



$$R = \begin{pmatrix} \varepsilon^{0} & & \\ \varepsilon^{-1} & \varepsilon^{0} & \\ \varepsilon^{-2} & \varepsilon^{-1} & \varepsilon^{0} \end{pmatrix} \begin{pmatrix} \mathbb{1} & & \\ \varepsilon^{0} & \mathbb{1} & \\ \varepsilon^{-1} & \varepsilon^{0} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & & \\ & \mathbb{1} & \\ \varepsilon^{0} & & \mathbb{1} \end{pmatrix}$$

Entries of R defined by polynomial differential constraints that are coupled…
…but systematically solvable, at worst as series expansions

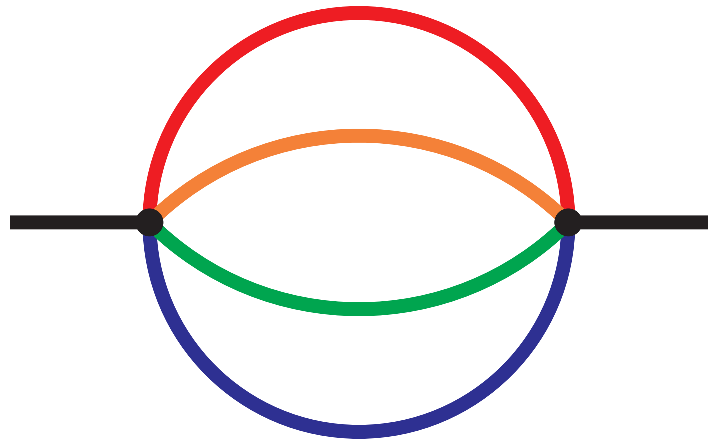
Gives a complete algorithm for deriving ε-factorized differential equations for Feynman integrals

Applications

Application

Three-loop Banana Integral

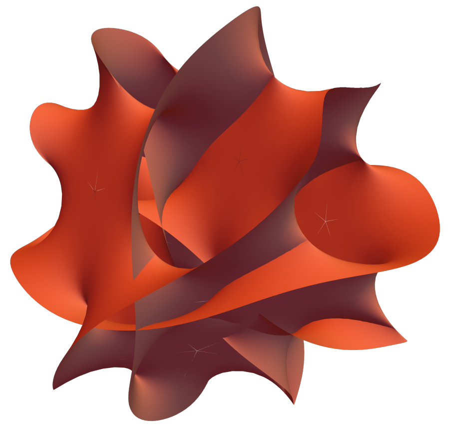
[SP, Teschke, Weinzierl, Wang, 2507.23594]



Four parameters

$$y_i = -\frac{m_i^2}{p^2}$$

Associated
K3 surface
(CY 2-fold)



Twist

$$u = P_1^{\varepsilon} P_2^{\varepsilon} P_3^{-1/2-\varepsilon} P_4^{-1/2-\varepsilon} P_5^{-1/2-\varepsilon}$$

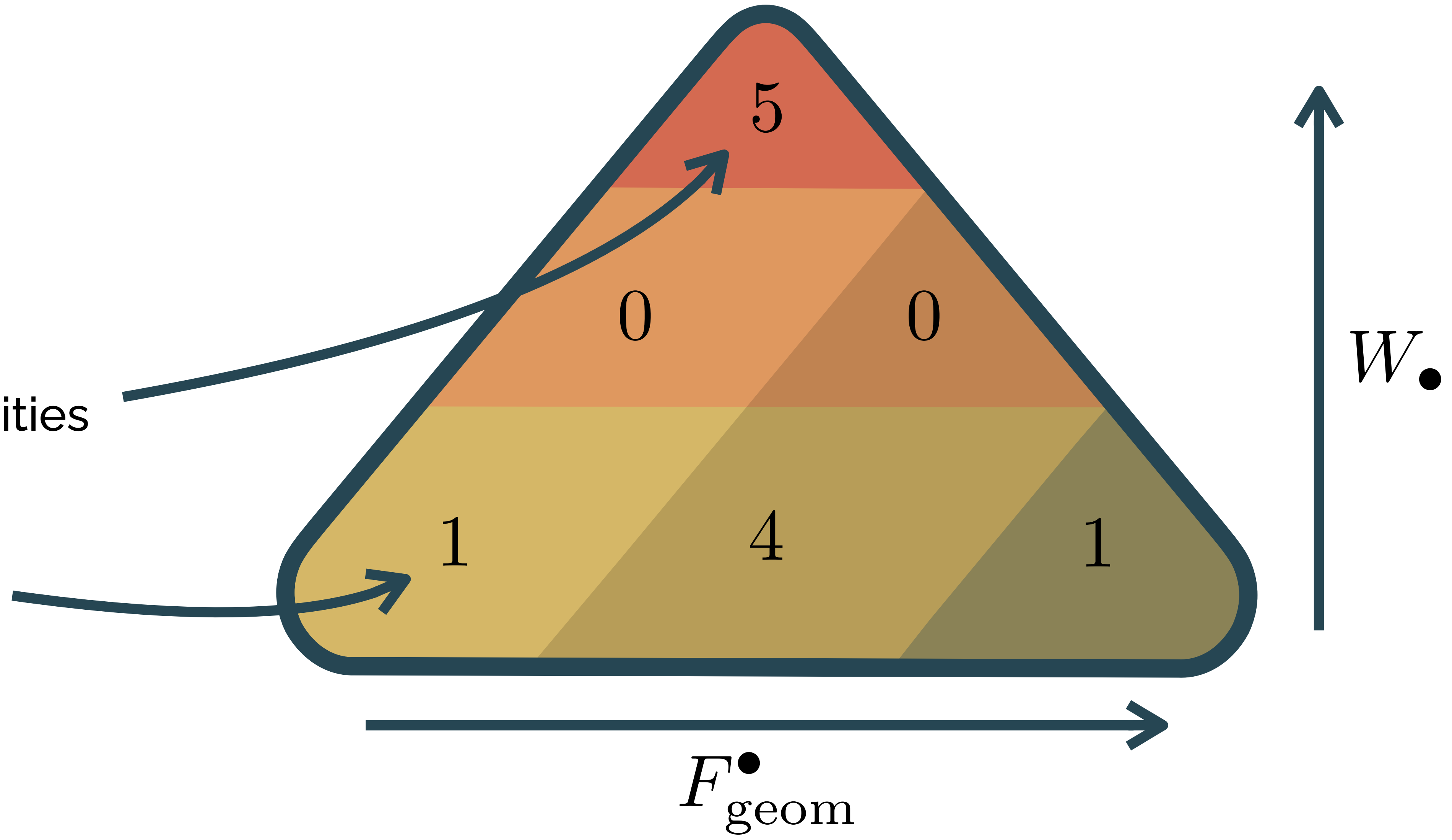
Baikov polynomials

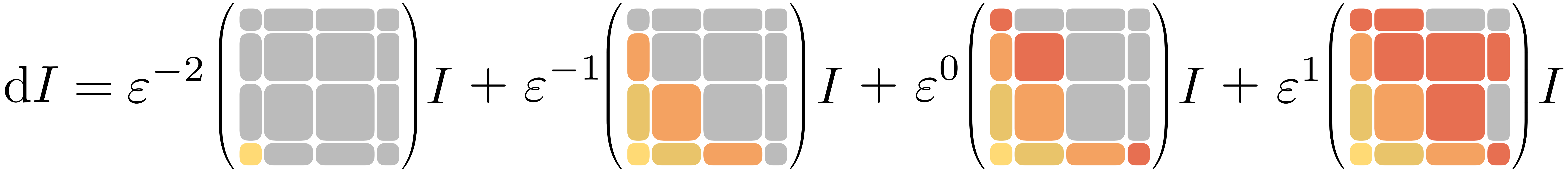
$$\left.\begin{aligned} P_1 &= z_1 \\ P_2 &= z_2 \end{aligned}\right\}$$

Options for log singularities

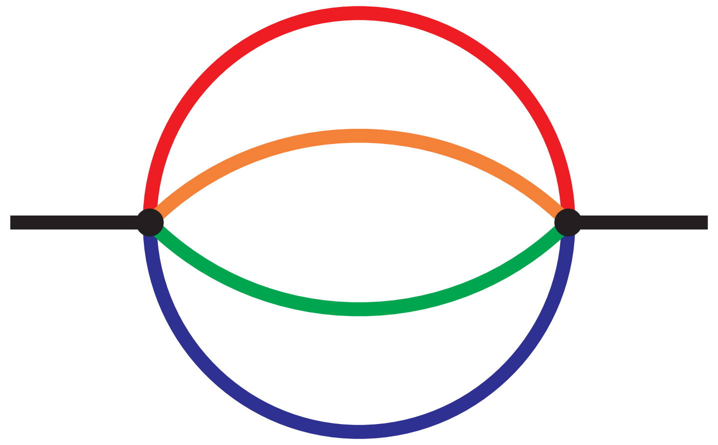
$$\left.\begin{aligned} P_3 &= z_2^2 - 2\left(1-y_1\right) z_2 + \left(1+y_1\right)^2 \\ P_4 &= \left(z_2-z_1\right)^2 + 2y_2\left(z_1+z_2\right) + y_2^2 \\ P_5 &= z_1^2 + 2\left(y_3+y_4\right) z_1 + \left(y_3-y_4\right)^2 \end{aligned}\right\}$$

Define K3 surface

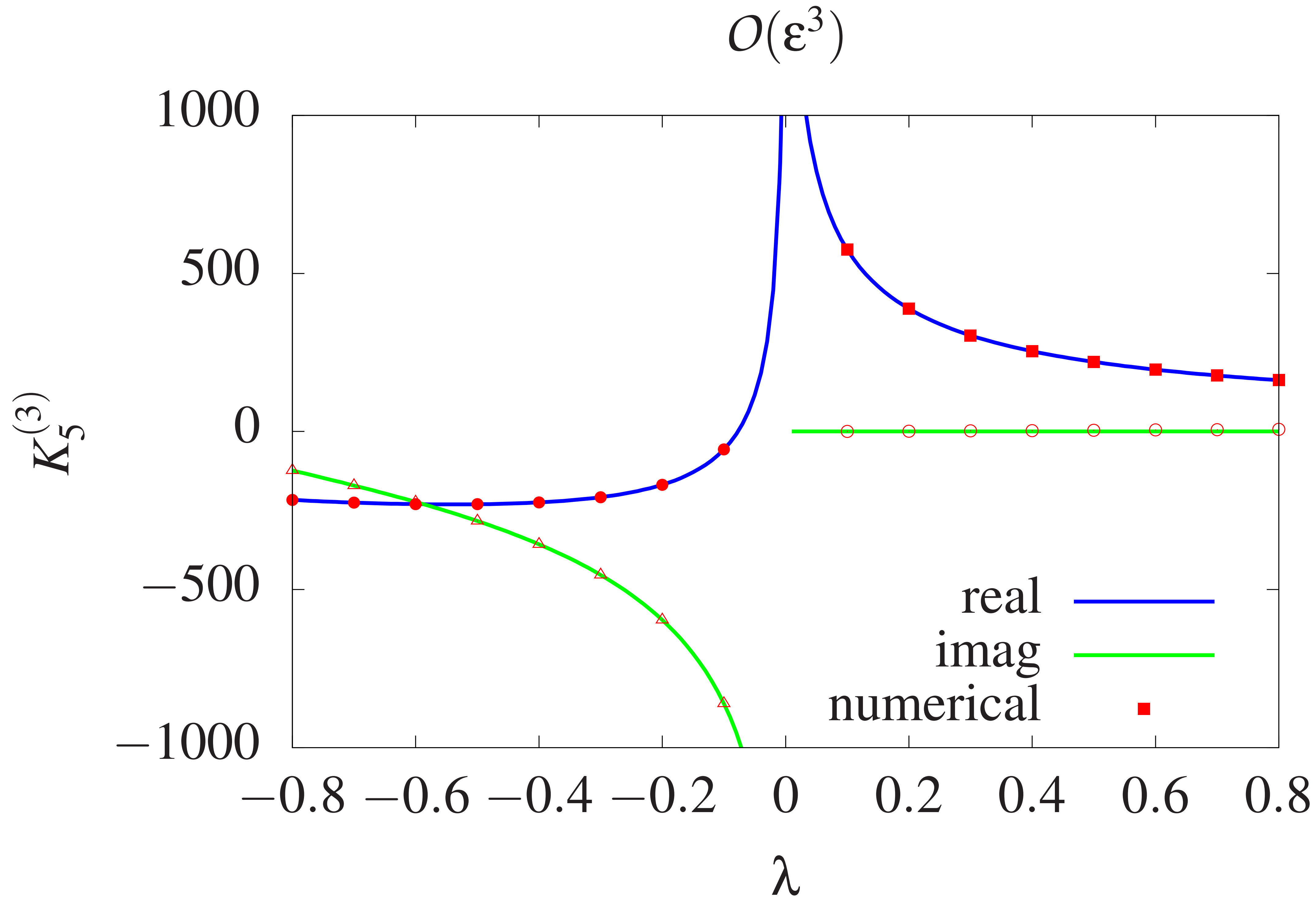


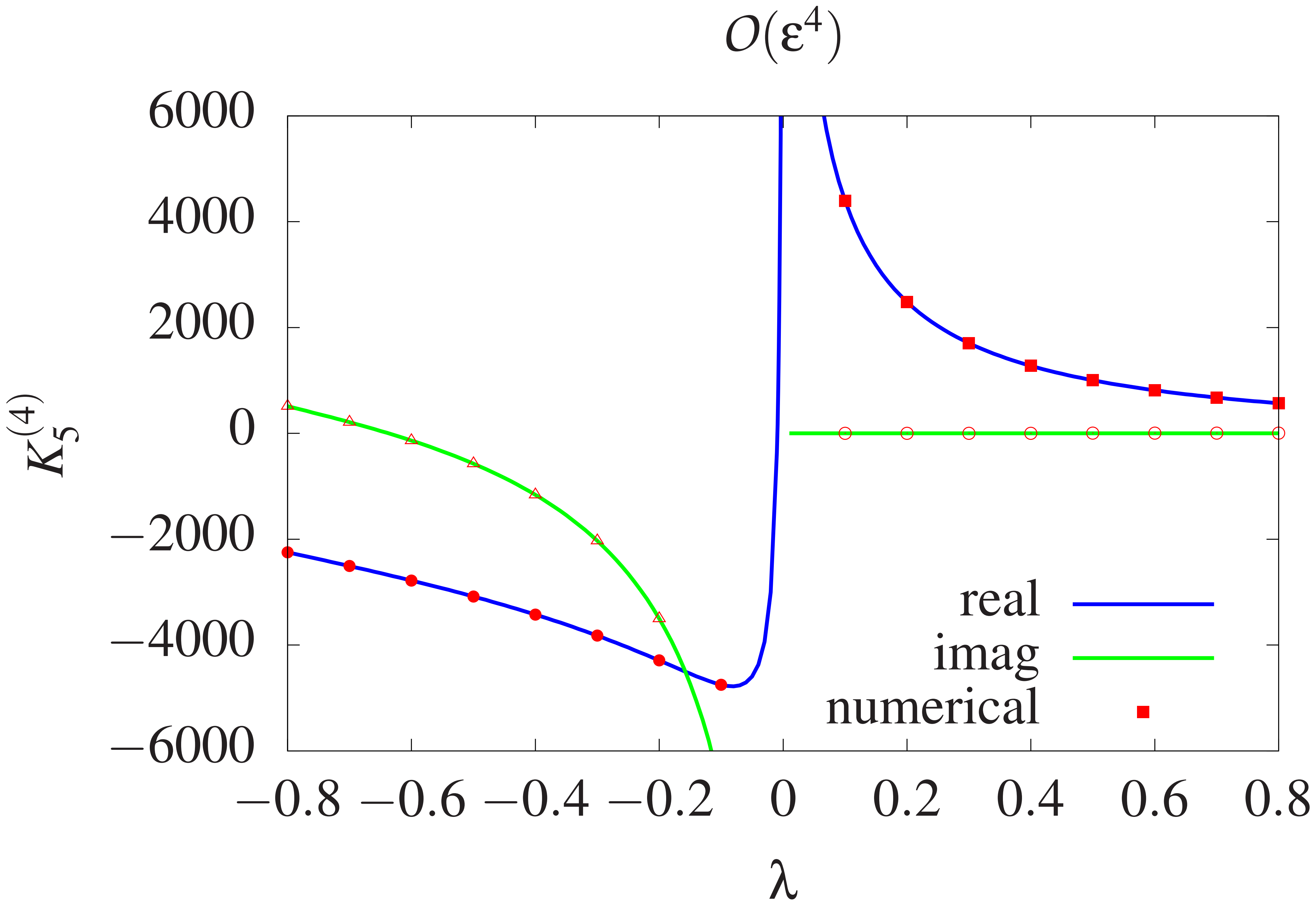


Allows high-precision numerical evaluation for



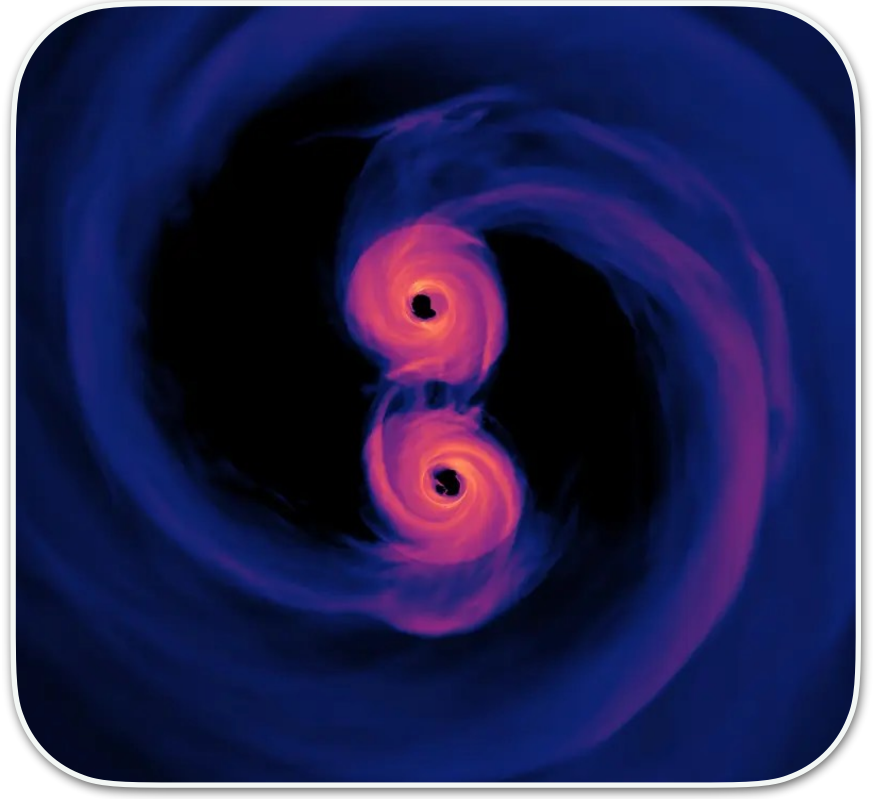
“Random” parameter slice $\lambda = \dfrac{(m_W + m_Z + m_H + m_t)^2}{-p^2}$





Conclusions

- Developed general algorithm derive ε-factorized differential equations for Feynman integrals
  - No case-by-case geometric analysis required
  - No requirement of making “good” choices by user
  - Allows deriving solutions as multi-variate series expansions
- Applied to non-trivial Calabi—Yau example (three-loop Banana integral), verified against a number of known examples



- Machinery ready to be applied to gravity integrals
  - Automatically handles new complicated geometries
  - Automatically makes good choices even for trivial geometries

Thank you!