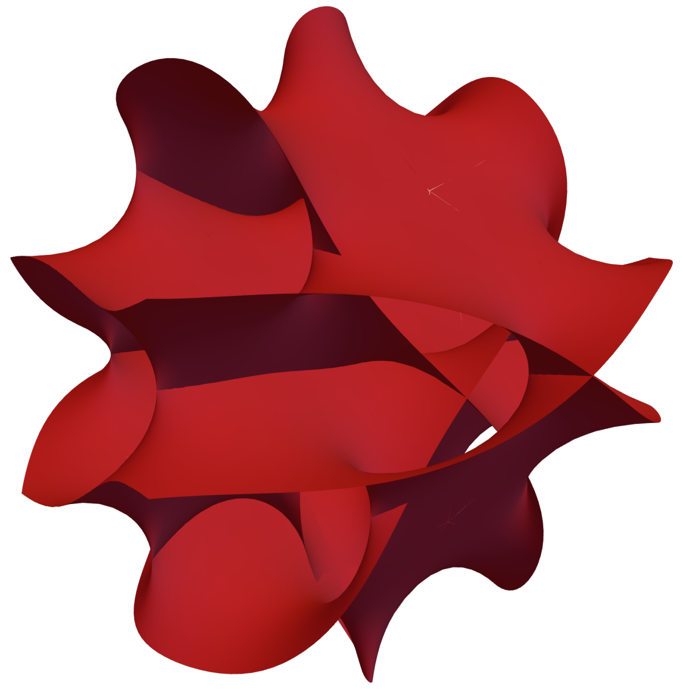
Geometry in Feynman Integrals

Of Calabi—Yaus and Higher-Genus Curves



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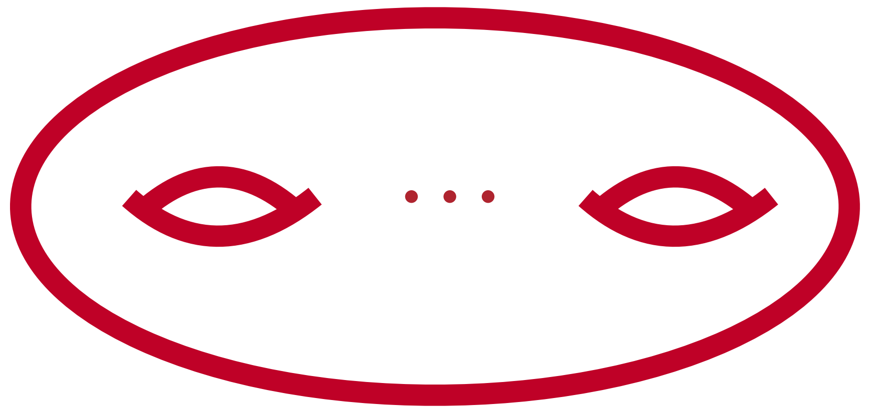
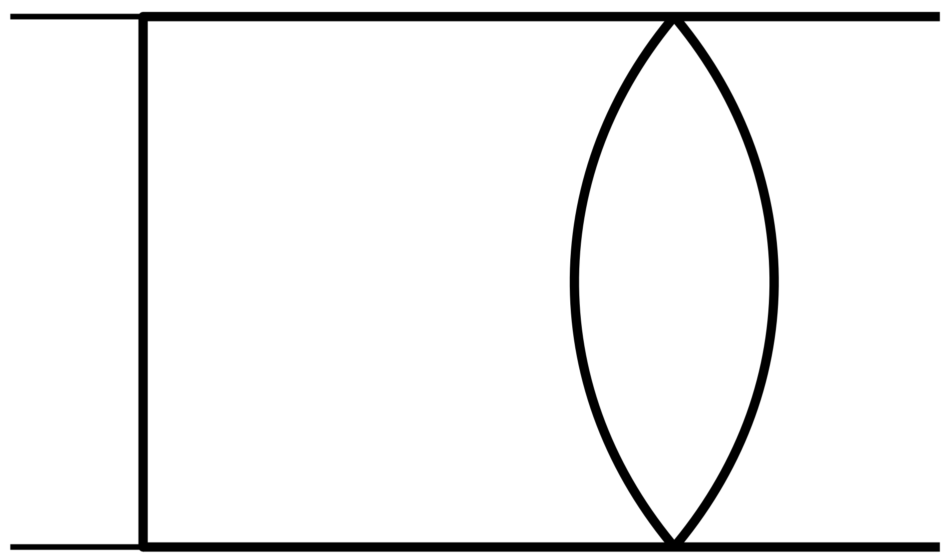
Based on work in collaboration with Xing Wang and Stefan Weinzierl
`2207.12893` (JHEP 09 (2022) 062)
`2211.04292` (PRL 130 (2023) 10, 101601)
`2212.08908` (JHEP 04 (2023) 117)



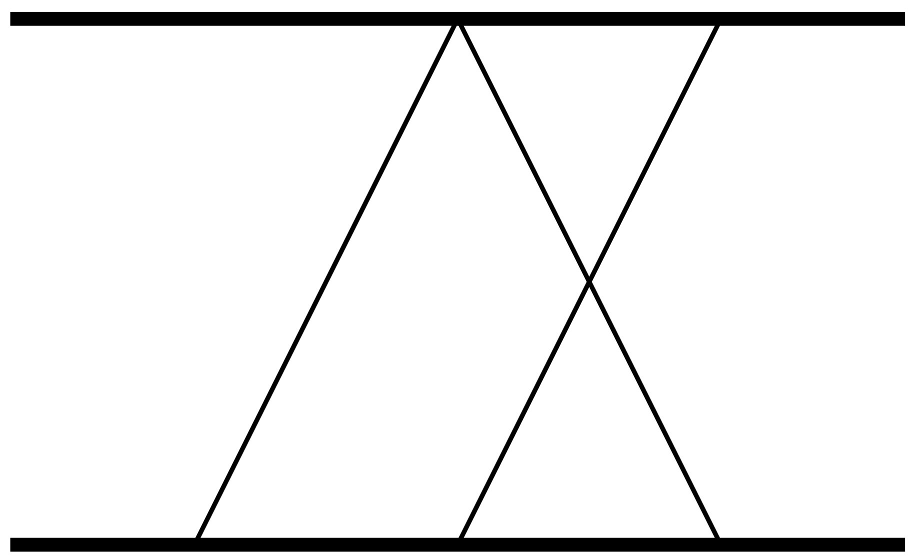
as well as Ben Page, Andrew McLeod, Robin Marzucca, and Stefan Weinzierl
`2307.11497`

Feynman Integrals

QCD



Gravity



Theory independent building blocks capturing most loop-level information

Boil em,
mash em,
stick em in an amplitude

Integrals associated to geometries
Determines suitable function space

Sphere ↓ **MPLs**

Elliptic curve ↓ **Elliptic Integrals, modular forms, EMPLs**

. . .

What is there beyond elliptics?

Feynman Integral Evaluation
A How To

1. List all integrals appearing in your problem
2. Use identities (integration-by-parts, symmetries, etc.) to obtain basis integrals $\vec{I}$ ("Master Integrals")
3. Write down a differential equation $\mathrm{d}\vec{I} = A\vec{I}$
4. Solve differential equation (???)
5. Obtain expressions for Laurent series in ε of $\vec{I}$

Trick: Choose Master Integrals such that

$$dI = \varepsilon A I$$ [Henn '13]

Find basis and variables, such that

- A independent of ε
- A consisting of functions we "understand well"

Analytic understanding and/or fast numerical evaluation

Given boundary value I_0

Can then trivially evaluated at any order in ε: $I = \mathbb{P}\exp\left(\varepsilon \int A\right) I_0$

Geometry associated to integral determines space of forms in A

Fantastic Geometries

and where to find them

How do we identify geometry of integrals?

Graph Polynomial

$$I \sim \int \prod \mathrm{d}\alpha_i \alpha_i^{\nu_i - 1} \frac{U^{\nu-(l+1)D/2}}{F^{\nu - lD/2}}$$

U/F **define projective variety**

Extracting geometry is hard!

Maximal Cuts

$$\mathrm{MaxCut}(I) \sim \int \frac{\prod \mathrm{d}\ell_j^D}{\prod_i D_i} /.\{D_i \to \delta(D_i)\}$$

Homogeneous solution to differential equation of full integral

Skeletonized version of integral

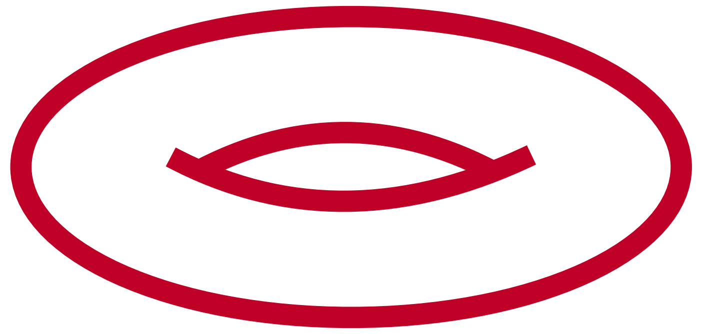
Much simpler to extract geometry

How do we generalize from the elliptic integrals?

Elliptic Feynman integrals are phenomenological state of the art

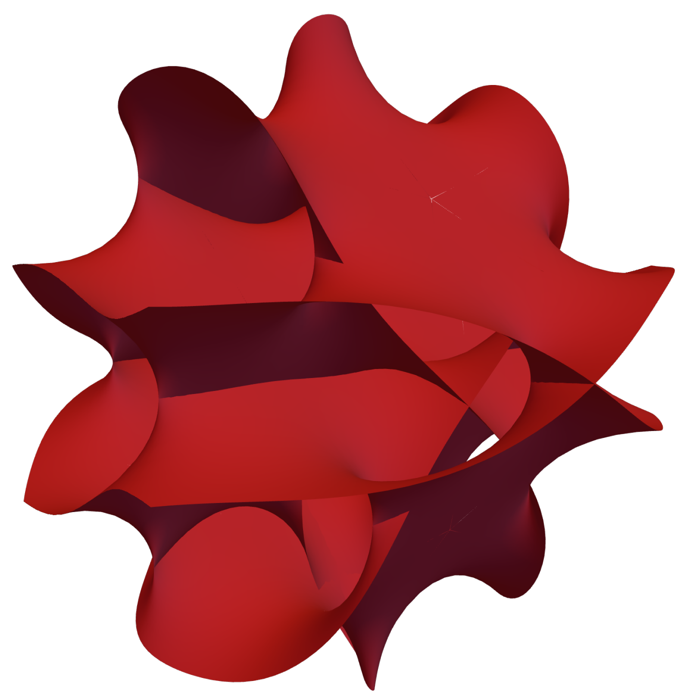
What else is there?

Elliptic Curve



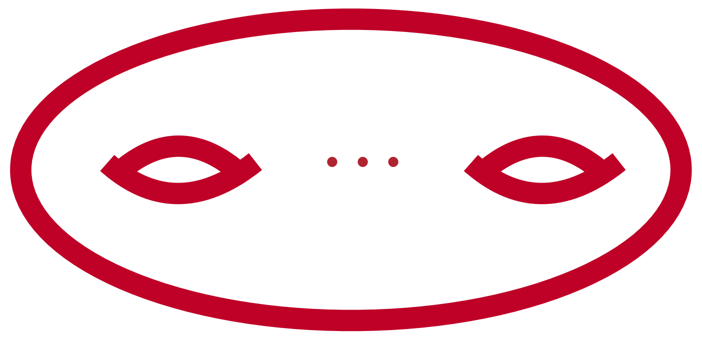
Dimension
1

Genus
1



Higher dimensional varieties

Curves of higher genus



e.g. Calabi—Yaus

e.g. Hyperelliptic curves

Part 1
Calabi–Yaus

Calabi—Yaus in Feynman Integrals

Compute maximal cut and takes as many residues as possible

$$\mathrm{MaxCut}\, I \sim \int \frac{\mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n}{\sqrt{P(\alpha_1, \dots, \alpha_n)}}$$

Calabi—Yau n-fold

Hypersurface in weighted projective space

[Bourjaily, McLeod, Vergu, Volk, von Hippel, Wilhelm, '20]

$$[1 : \alpha_1 : \dots : \alpha_n : y] \in \mathbb{WP}^{1,1,\dots,1,(n+1)}$$

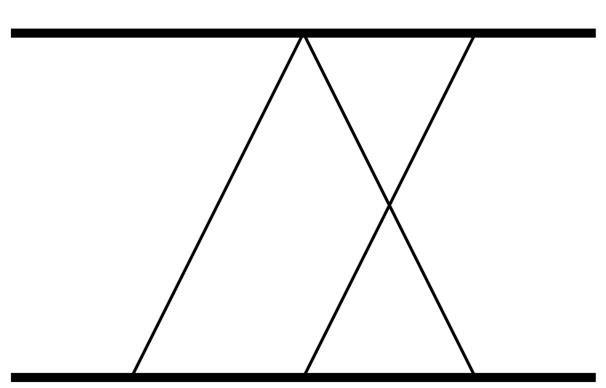
$$y^2 = P(\alpha_1, \dots, \alpha_n) \quad \text{with} \quad \deg P = 2(n+1)$$

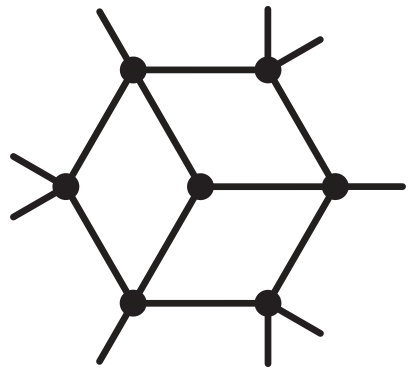
Codimension 1 = Dimension n

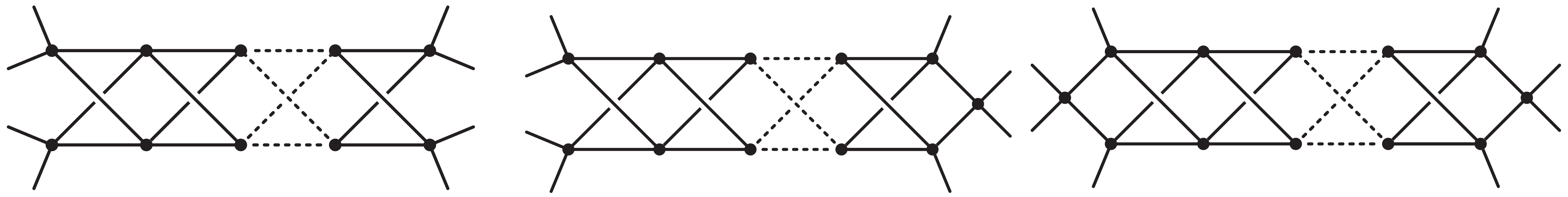
$\mathrm{MaxCut}(I)$ **is a so-called period of the Calabi—Yau**

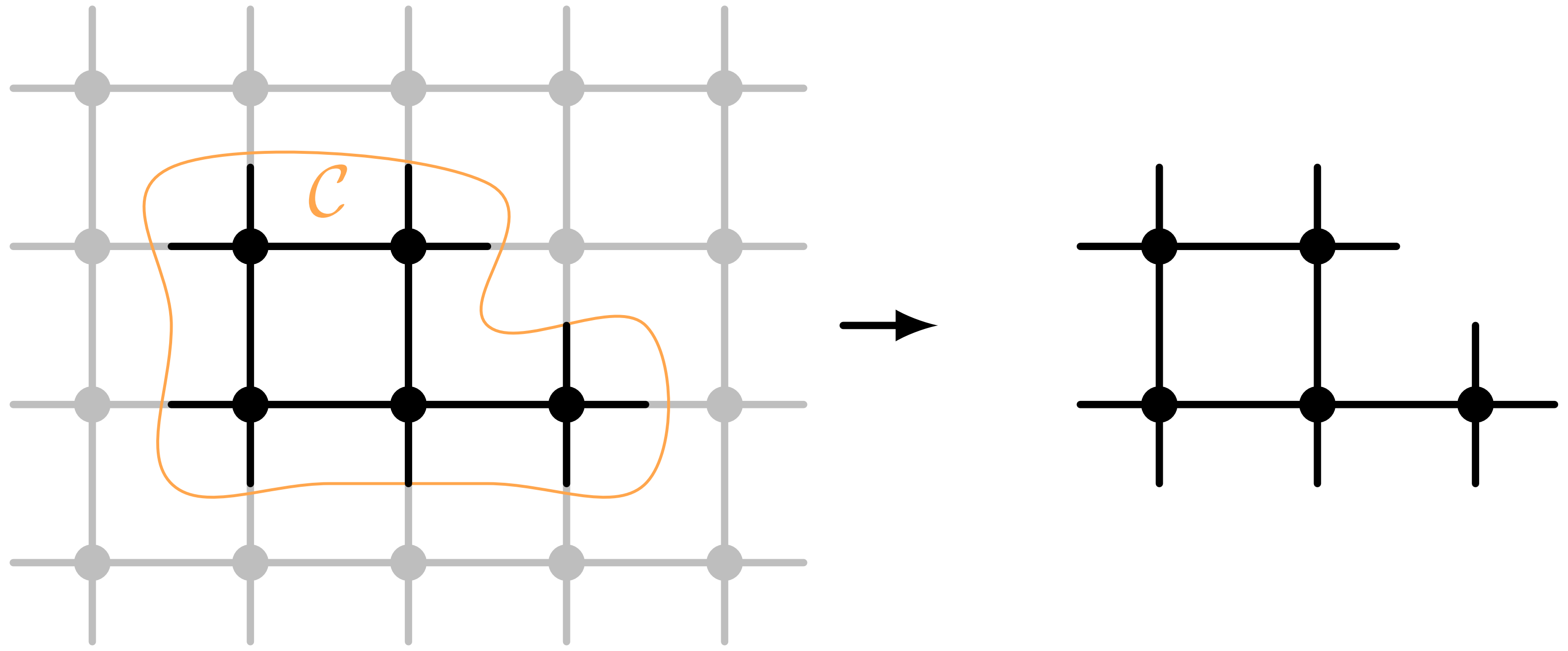
Calabi–Yaus: “A (bounded) bestiary”

[Bourjaily, McLeod, von Hippel, Wilhelm, ’19]
[Duhr, Klemm, Loebbert, Nega, Porkert, Tancredi, ’22, ’23]





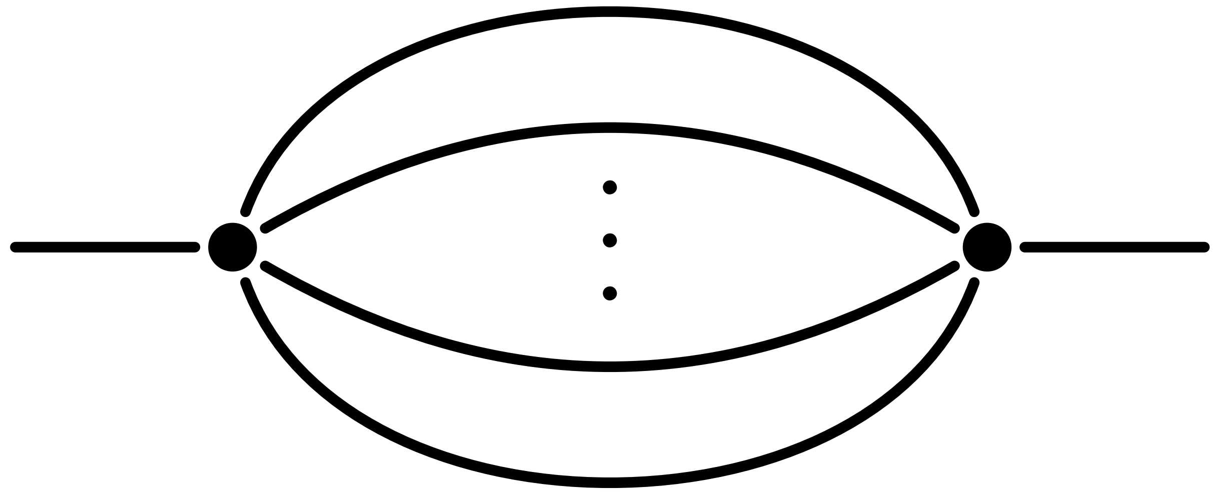




and more…

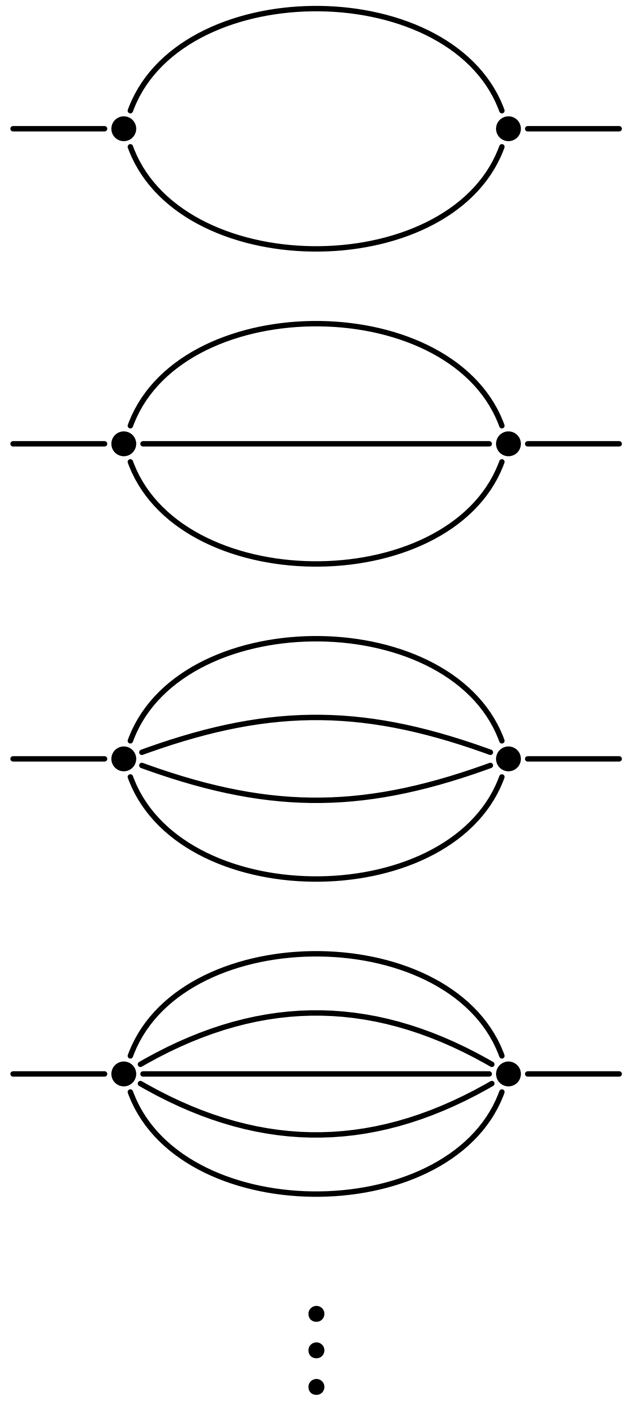
Simplest Example: Banana Integrals

in $D = 2 - 2\varepsilon$



Bananas: A Calabi–Yau Prototype

Calabi–Yau…



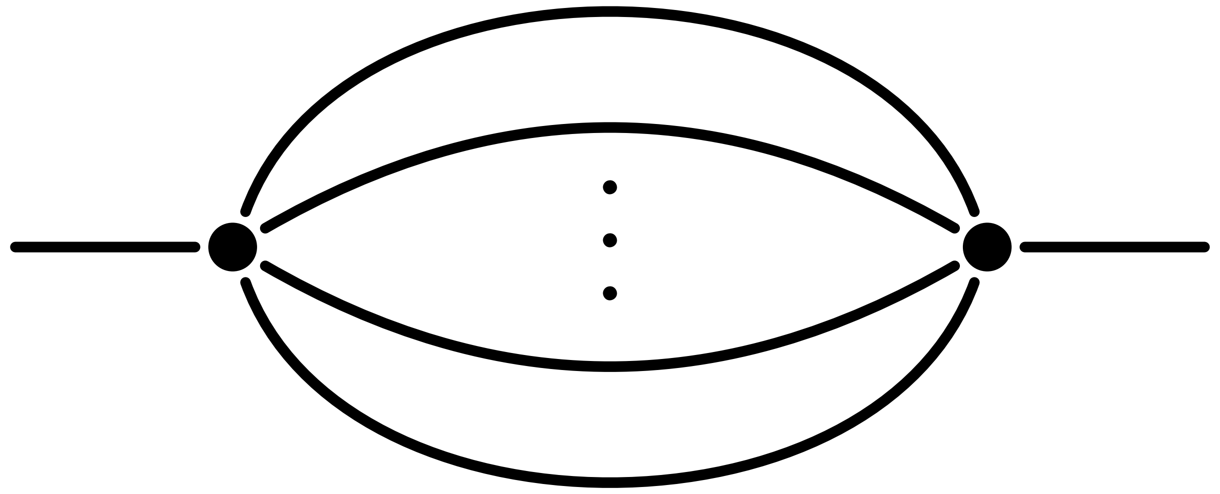
0-fold

1-fold = elliptic curve
“Sunrise”

2-fold = K3 surface

3-fold

⋮



ℓ-loop Banana integral

$\hat{=}$

$(\ell - 1)$-fold Calabi–Yau manifold

ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]

Simplification: Equal-mass $\rightarrow$ single scale

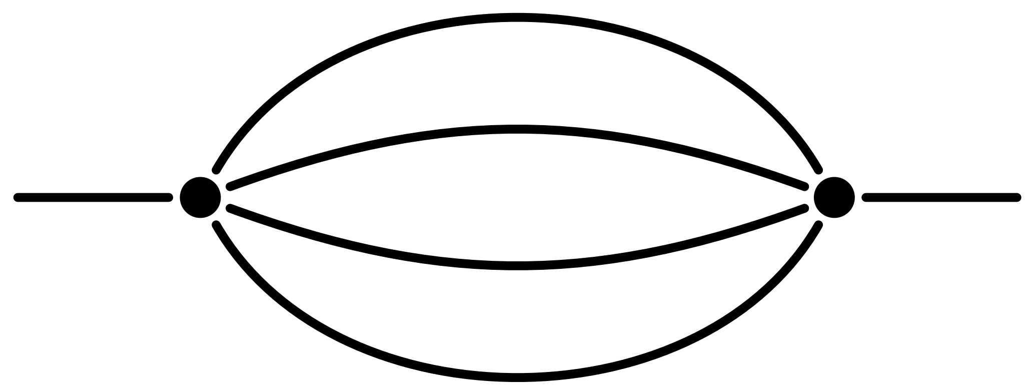
Kinematic variable

$$x = \frac{p^2}{m^2} \qquad y = -\frac{m^2}{p^2}$$

“Trivial” Calabi–Yaus

***Essentially* elliptic**

Three-loop Banana



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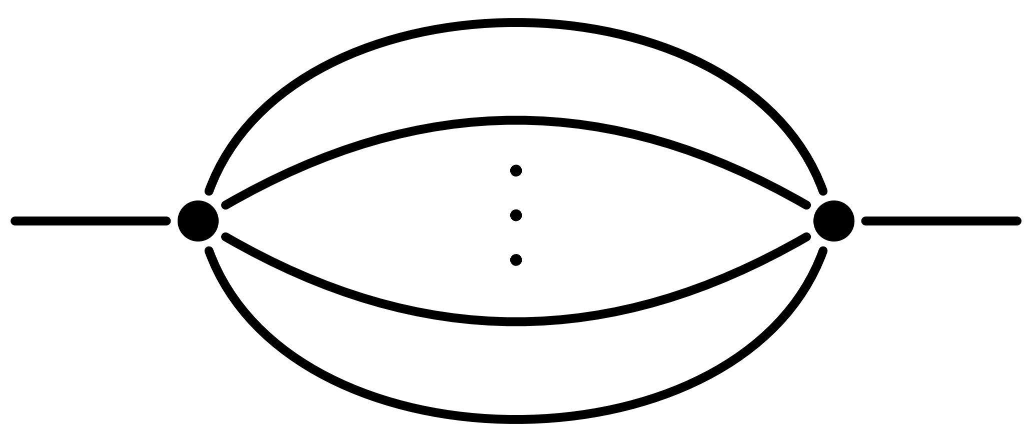
“Non-trivial” Calabi–Yaus

Non-elliptic

(≥Four)-loop Banana



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The Three-Loop Banana Integral

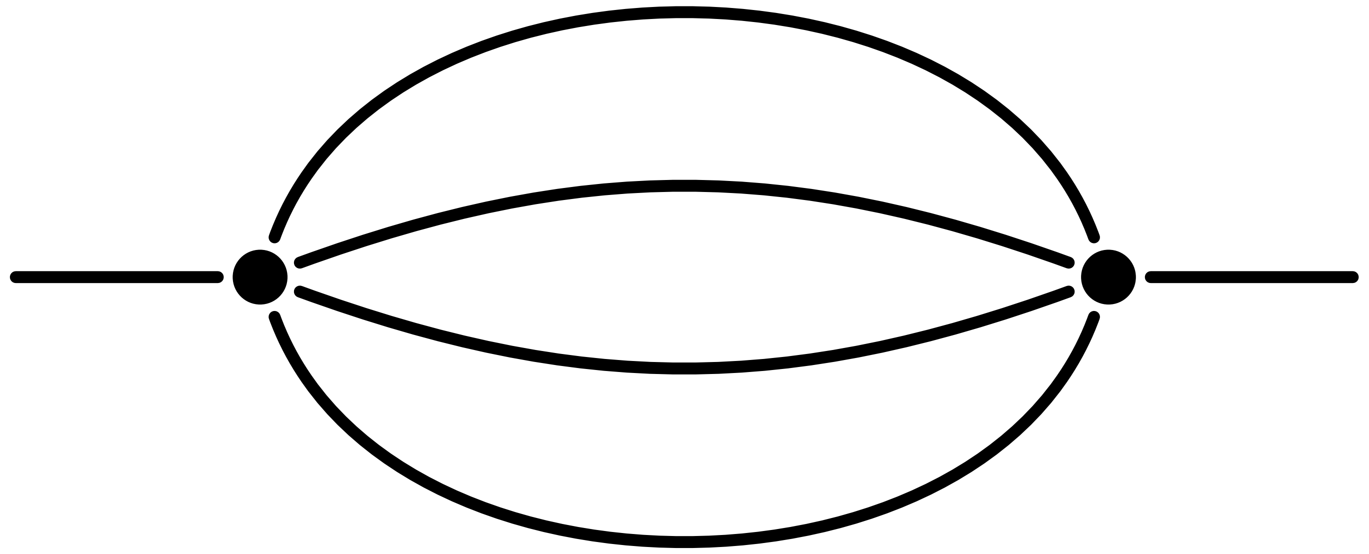
Simplest example of Feynman integral beyond elliptic:

Calabi–Yau 2-fold

Equal-mass case: closely connected to sunrise integral

Extensively studied in the past:

- Leading term in ε [Bloch, Kerr, Vanhove, 14']
- ε-factorized form [Primo, Tancredi,17']
- Master integrals in $d=2$ in terms of eMPLs $\tilde{\Gamma}$ [Broedel, Duhr, Dulat, Marzucca, Penante, 19']
- DEQ with meromorphic modular forms [Broedel, Duhr, Matthes, 21']
- ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]



Singularities:

$$x = \frac{p^2}{m^2} = 0,4,16,\infty$$

Picard-Fuchs Differential Operator

Annihilates $\mathrm{MaxCut}(I)$ (periods of Calabi–Yau)

→ Defines geometry

3-loop banana in $D=2$:

$$\mathcal{L}_3^{(0)} = \frac{d^3}{dx^3} + \left[\frac{3}{x} + \frac{3}{2(x-4)} + \frac{3}{2(x-16)}\right]\frac{d^2}{dx^2} + \frac{7x^2-68x+64}{x^2(x-4)(x-16)}\frac{d}{dx} + \frac{1}{x^2(x-16)}.$$

with solutions $\mathcal{L}_3^{(0)}\,\omega_i = 0$ where $\omega_i = \mathrm{MaxCut}(I_{1111})|_{\gamma_i}$ on three independent contours γ_i

$\mathcal{L}_3^{(0)}$ is a symmetric square

[Verrill, 96'; Joyce, 72']

There exists an operator

$$\mathcal{L}_2^{(0)} = \frac{d^2}{dx^2} + \left[\frac{1}{x} + \frac{1}{2(x-4)} + \frac{1}{2(x-16)}\right]\frac{d}{dx} + \frac{(x-8)}{4x(x-4)(x-16)}$$

← Sunrise in disguise

with solutions $\psi_1,\ \psi_2,\ \mathcal{L}_2^{(0)}\,\psi_i = 0$ such that

$$\omega_i \in \langle \psi_1^2,\ \psi_1\psi_2,\ \psi_2^2 \rangle$$

ε-Factorization: Sunrise

Make the ansatz

$$I_1 = \varepsilon^2 I_{110},$$
$$I_2 = \varepsilon^2 \frac{\pi}{\psi_1} I_{111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$

$$\tau = \frac{\psi_2}{\psi_1}$$

Periods of elliptic curve

$$\mathrm{d}I = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & 1 \\ \eta_3 & \eta_4 & \eta_2 \end{pmatrix} I$$

A organised by modular weight

η_k : Modular forms of $\Gamma_1(6)$ of weight k, independent of ε

✓ A independent of ε

✓ A consists of modular forms

"well understood"

ε-Factorization: Three-loop Ansatz

Make the ansatz

$$I_1 = \varepsilon^3 I_{1110},$$
$$I_2 = \varepsilon^3 \frac{1}{\omega} I_{1111},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$
$$I_4 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3.$$

No assumptions for ω and τ required

↓

$$dI = \tilde{A} I$$

Requiring $\tilde{A} = \varepsilon A$ → constraints on ω, J, F_{32}, F_{42}, F_{43}

J: $\frac{\mathrm{d}x}{\mathrm{d}\tau}$

Eliminate Non-ε-Factorized Pieces

Already ε-factorized

$$dI = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_{2,2} & 1 & 0 \\ 0 & \tilde{A}_{3,2} & \tilde{A}_{3,3} & 1 \\ \tilde{A}_{4,1} & \tilde{A}_{4,2} & \tilde{A}_{4,3} & \tilde{A}_{4,4} \end{pmatrix} I$$

$\tilde{A}_{4,k}$ contains term ε^{-4+k} through ε

Five variables, six constraints

$\omega, J, F_{32}, F_{42}, F_{43}$

$\rightarrow$ **One non-trivial constraint!**

Periods of elliptic curve

Satisfied for $\omega = (x\psi_1^{\text{sun}})^2 \quad \tau = \dfrac{\psi_2^{\text{sun}}}{\psi_1^{\text{sun}}}$

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = (2\pi i)\varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -f_{2,a} - f_{2,b} & 1 & 0 \\ 0 & f_{4,b} & -f_{2,a} + 2f_{2,b} & 1 \\ f_{4,a} & f_6 & f_{4,b} & -f_{2,a} - f_{2,b} \end{pmatrix} I$$

Constraints allow symmetry

Alphabet: $\mathcal{A} = \{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_6\}$.

Function space of Alphabet

Meromorphic modular forms + Special function F_2

Iterated integral of meromorphic modular form of weight 6

$F_2 = I(1, g_6; \tau)$ $g_6 = \frac{x(x-8)(x+8)^3}{864(4-x)^{\frac{3}{2}}(16-x)^{\frac{3}{2}}} \left(\frac{\psi_1}{\pi}\right)^6$

$$I_2 = \varepsilon^3 \left(\frac{4}{3}\zeta_3 + I(1,1,f_{4,a};\tau) \right) + \mathcal{O}(\epsilon^4)$$

Obtained expressions for all masters up to ε^6

Numerics via q-expansion

“Trivial” Calabi–Yau Summary

ε**-factorized form:** Ansatz, then solve constraints algorithmically

Symmetric square: Three-loop banana integral related to elliptic curve

Function space: Meromorphic modular forms, plus iterated integrals thereof (F_2)

Expectation: This generalizes beyond the banana!

Single scale +
2-fold Calabi–Yau integral +
degree 3 Picard–Fuchs operator

→

“trivial” Calabi–Yau
i.e. symmetric power
of elliptic curve

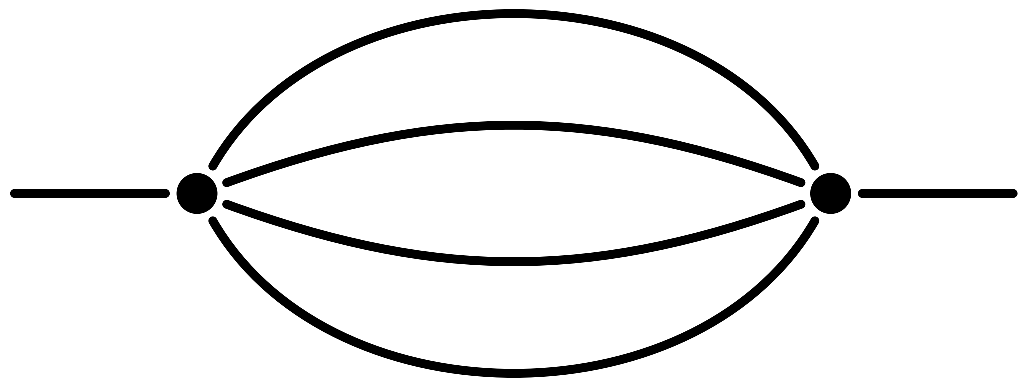
→

ε-factorised DEQ from Ansatz
Function space
essentially elliptic

“Trivial” Calabi–Yaus

***Essentially* elliptic**

Three-loop Banana



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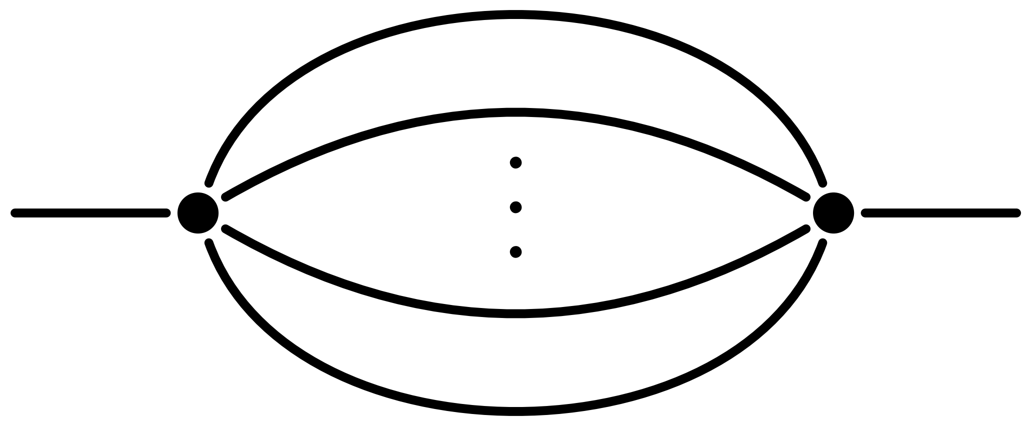
“Non-trivial” Calabi–Yaus

Non-elliptic

(≥Four)-loop Banana



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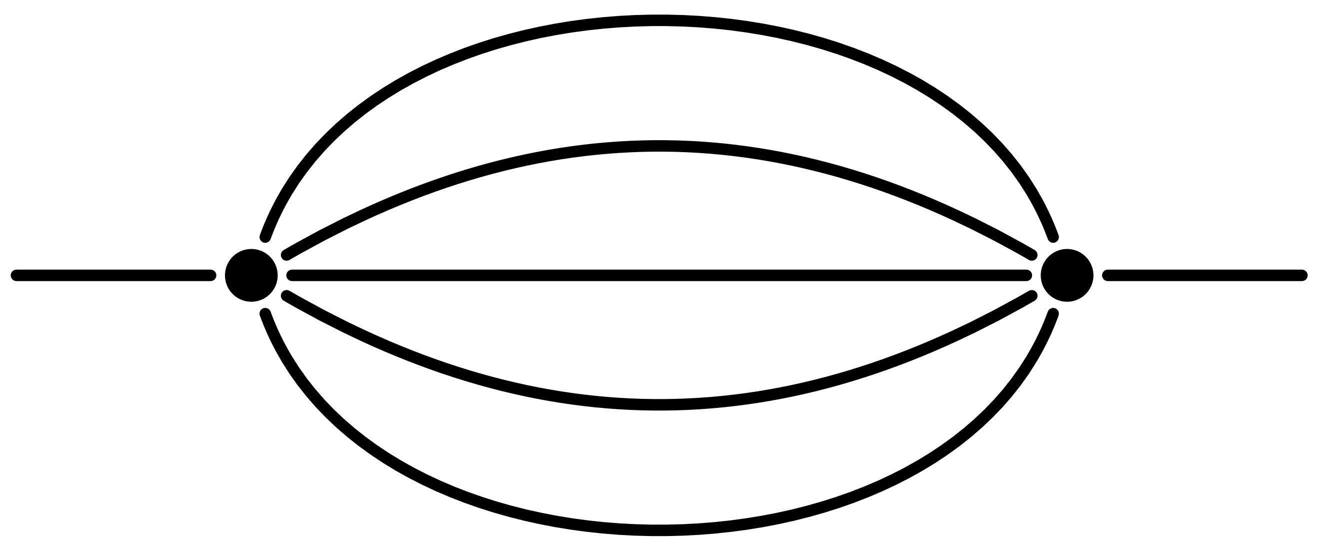
The Four-Loop Banana Integral

First banana integral with “non-trivial” Calabi–Yau:

Not related to elliptic curves



Integral already studied in the past

ℓ-loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]

Singularities:

$y = -\frac{m^2}{p^2} = 0, -1, -\frac{1}{9}, -\frac{1}{25}, \infty$

Algebraic Variety from graph polynomial
Hypersurface in $\mathbb{CP}^4$ with

$$1/y = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} + \frac{1}{\alpha_5}\right)$$

Calabi–Yau very well known

Studied in [Hulek, Verrill, 05’; …]
Known as `AESZ34` [Almkvist, van Enckevort, van Straten, Zudilin]

ε-Factorization: Four-loop Ansatz

Guess the pattern?

$$
\begin{aligned}
I_1 &= \varepsilon^4 I_{11110}, \\
I_2 &= \varepsilon^4 \frac{1}{\omega} I_{11111}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.
\end{aligned}
$$

?

$dI = \varepsilon A I$ leads to inconsistent constraints!
$\rightarrow$ No solution!

ε-Factorization: Four-loop Ansatz (fixed)

Modify ansatz!

$$
\begin{aligned}
I_1 &= \varepsilon^4 I_{11110}, \\
I_2 &= \varepsilon^4 \frac{1}{\omega} I_{11111}, \\
I_3 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2, \\
I_4 &= \frac{1}{\varepsilon}\boxed{\frac{1}{K_1}}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3 \\
I_5 &= \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4.
\end{aligned}
$$

ℓ-loop Banana Integrals define special Calabi–Yau manifolds

Picard–Fuchs operators are **Calabi–Yau operators**

[Almkvist, van Enckevort, van Straten, Zudilin, 05’]
[M. Bogner, 13’]

K_1 is a Y-invariant of Calabi–Yau operator

Start appearing at 3-fold

$dI = \varepsilon A I$ leads to consistent constraints!

No prior knowledge of K_1 required!

Fixed by constraints (up to rescaling)

What is the function space of "non-trivial" Calabi–Yaus to solve constraints?

Currently unknown

But for fast numerics, imitate elliptics:

q-expansion

Expansion point

$y = -m^2/p^2 = 0$ **(MUM-point)**

Frobenius basis:

$\omega_1, \omega_2, \omega_3, \omega_4$

Expansion coordinate:

$q = \exp(2\pi i \tau)$, $\tau = \omega_2/\omega_1$

Canonical variables for Calabi–Yau operators

Generalization of
τ (ratio of periods)
q (nome)
from elliptic case $\ell = 2$

Four-Loop solutions

$q(y) = \exp(2\pi i \omega_2/\omega_1)$	$y - 8y^2 + 92y^3 - 1288y^4 + 20398y^5 + \mathcal{O}(y^6)$
$\omega = \omega_1$	$q + 3q^2 + q^3 + 23q^4 - 101q^5 + \mathcal{O}(q^6)$
$K_1 = \mathrm{d}^2/\mathrm{d}\tau^2(\omega_3/\omega_1)$	$1 - q + 17q^2 - 253q^3 + 3345q^4 - 43751q^5 + \mathcal{O}(q^6)$
J	$q + 16q^2 + 108q^3 + 672q^4 + 2570q^5 + \mathcal{O}(q^6)$
F_{32}	$c_{32} + 8q - 32q^2 + 512q^3 - 5872q^4 + 70008q^5 + \mathcal{O}(q^6)$
F_{42}	$c_{42} + 8q - 240q^2 + 4816q^3 - 90448q^4 + 1444008q^5$ $+c_{32}(-9q + 176q^2 - 2956q^3 + 44568q^4 - 611106q^5)$ $+c_{32}^2(q - 16q^2 + 220q^3 - 2600q^4 + 30018q^5)$ $+\mathcal{O}(q^6)$
$\vdots$	

Predictable from just Picard–Fuchs operator (first four rows)

Need to solve constraints (rows F_{32}, F_{42}, …)

Remaining freedom c_{32}, c_{42}, etc.
→ can impose symmetry on A

Fast numerical evaluation
(Within convergence radius)

Five-, Six-, All-Loop Ansatz

$$I_1 = \varepsilon^\ell I_{1\dots10},$$
$$I_2 = \varepsilon^\ell \frac{1}{\omega} I_{1\dots1},$$
$$I_3 = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_2 + F_{32} I_2,$$
$$I_4 = \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_3 + F_{42} I_2 + F_{43} I_3$$
$$I_5 = \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_4 + F_{52} I_2 + F_{53} I_3 + F_{54} I_4$$
$$\vdots$$
$$I_{\ell-1} = \frac{1}{\varepsilon}\frac{1}{K_2}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-2} + \sum_{i=2}^{\ell-2} F_{\ell-1,i} I_i$$
$$I_\ell = \frac{1}{\varepsilon}\frac{1}{K_1}\frac{\mathrm{d}}{\mathrm{d}\tau} I_{\ell-1} + \sum_{i=2}^{\ell-1} F_{\ell,i} I_i$$
$$I_{\ell+1} = \frac{1}{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau} I_\ell + \sum_{i=2}^{\ell} F_{\ell+1,i} I_i$$

Checked up to seven loops

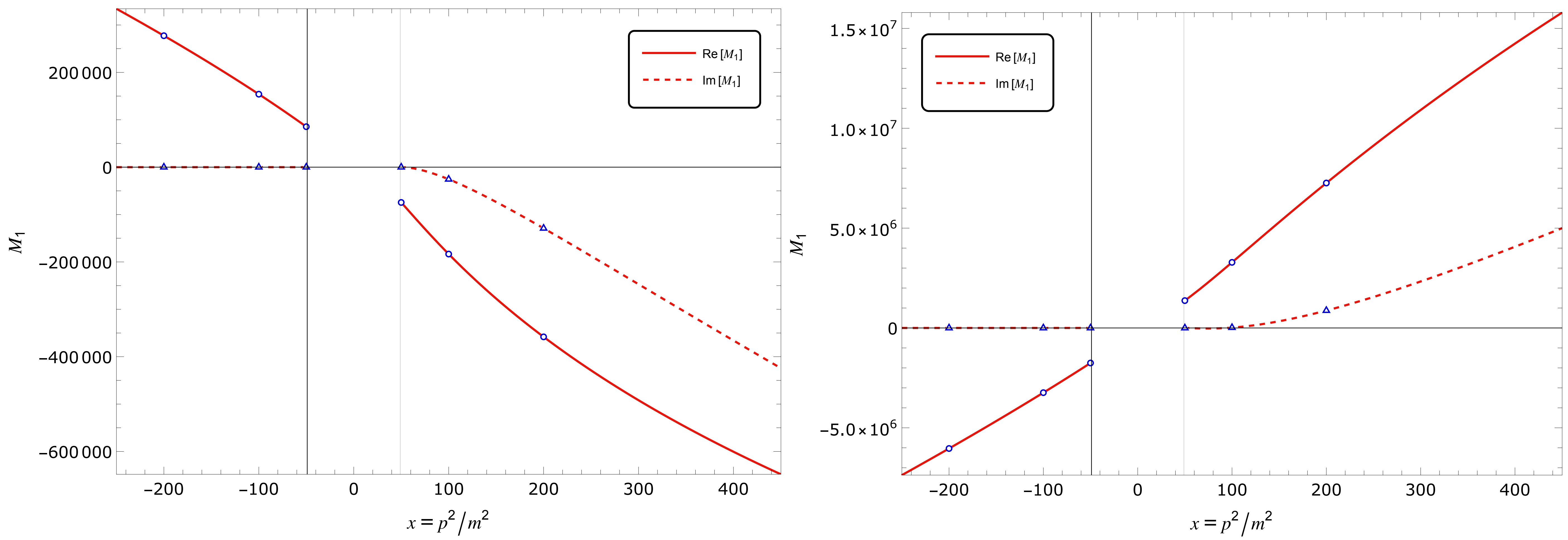
Ansatz with K_i being Y-invariants leads to consistent constraints

Checked up to six loops

Analytic expressions for Masters in terms of iterated integrals

$$I_2 = \left[I(1, K_1, K_2, K_1, 1, A_{71}; \tau) + \text{boundary} \right] \varepsilon^7 + \mathcal{O}(\varepsilon^8) \quad \text{etc.}$$

Numeric evaluation using q-expansion: agrees with `SecDec`



“Non-Trivial” Calabi–Yau Summary

ε-factorized form: Ansatz with information from Calabi–Yau operators
$\rightarrow$ Solve constraints algorithmically

Function space: currently unknown

Numerics: can obtain fast converging q-expansion

Expectation: Generalizes to other Calabi–Yau integrals

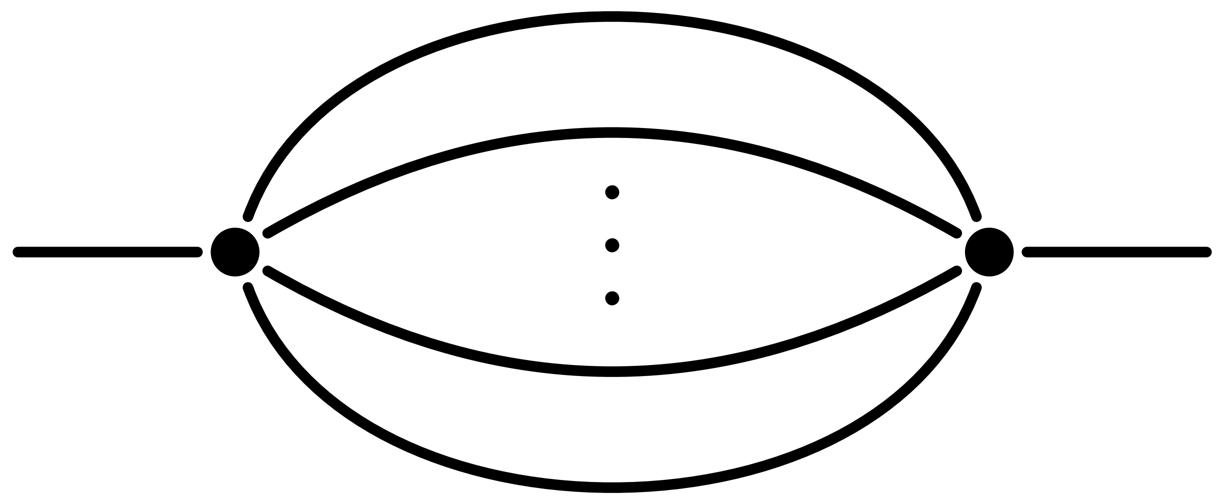
Single scale +
n-fold Calabi–Yau integral +
degree (n+1) Picard–Fuchs operator
$\rightarrow$
Calabi–Yau operator
“non-trivial” Calabi–Yau
$\rightarrow$
ε-factorised DEQ from Ansatz
Solve via series expansion

Part 1 Conclusions

Banana integrals: Simplest example of Calabi–Yau integrals

Simplification: Equal-mass = single scale

Single scale integral
n-fold Calabi–Yau,
degree (n+1) Picard–Fuchs operator



Ansatzing allows to find ε-factorised form algorithmically

Use information from theory of Calabi–Yau operators

Calabi-Yau 2-fold

Picard–Fuchs is symmetric square of elliptic curve

Modular forms

Calabi-Yau ($\geq$3)-fold

Not relatable to elliptics
Function space unknown

q-expansion

Part 2
Higher-Genus Curves

Algebraic Curves

Algebraic Curve: $f \in \mathbb{C}[y, z]$ $\quad y, z \in \mathbb{C}$ **such that** $f(y, z) = 0$

Simplest Example:

Elliptic Curves

$$f(y, z) = y^2 - (z - a_1)(z - a_2)(z - a_3)(z - a_4) = 0$$

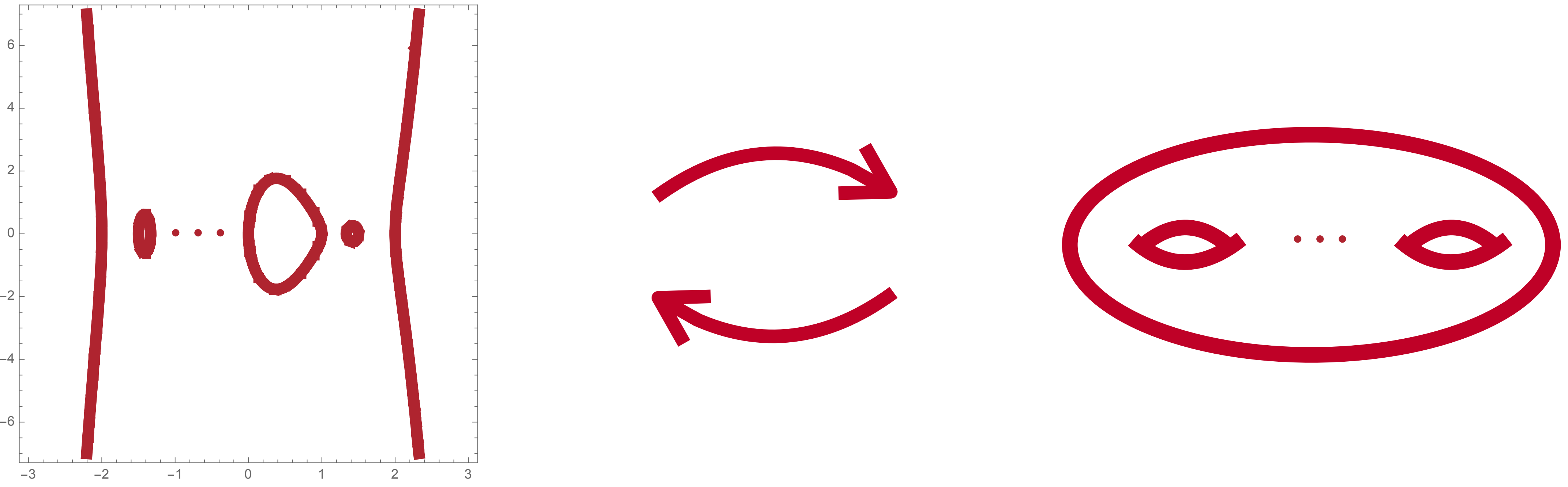
Genus 1

Next-to-Simplest Example:

Hyperelliptic Curves

$$f(y, z) = y^2 - (z - a_1) \ldots (z - a_{2g+2}) = y^2 - P_{2g+2}(z) = 0$$

Genus g

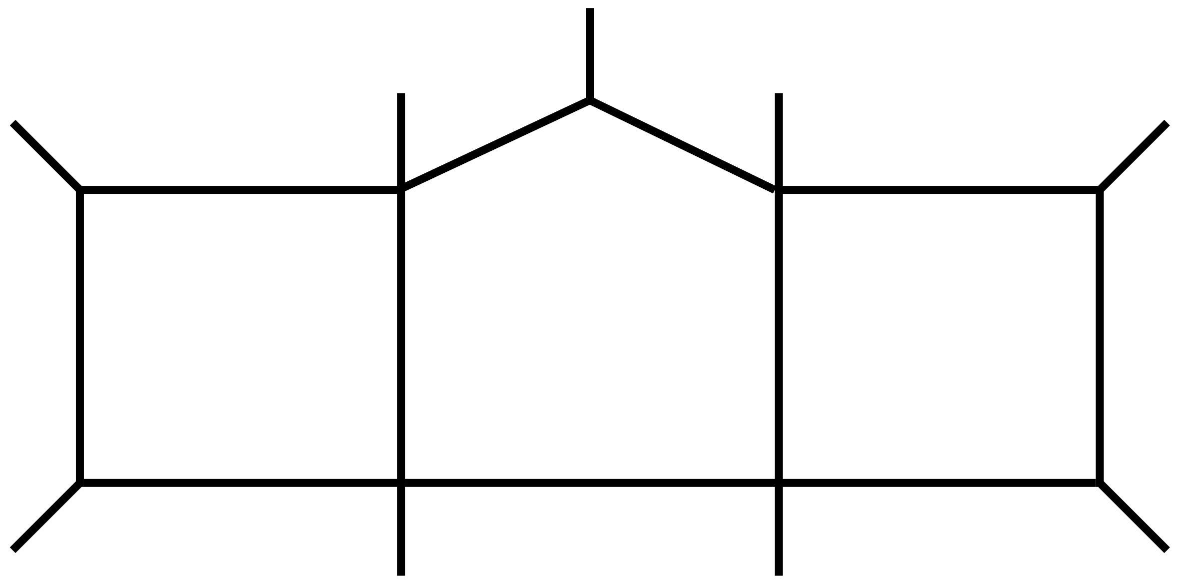


Higher-Genus Feynman Integrals

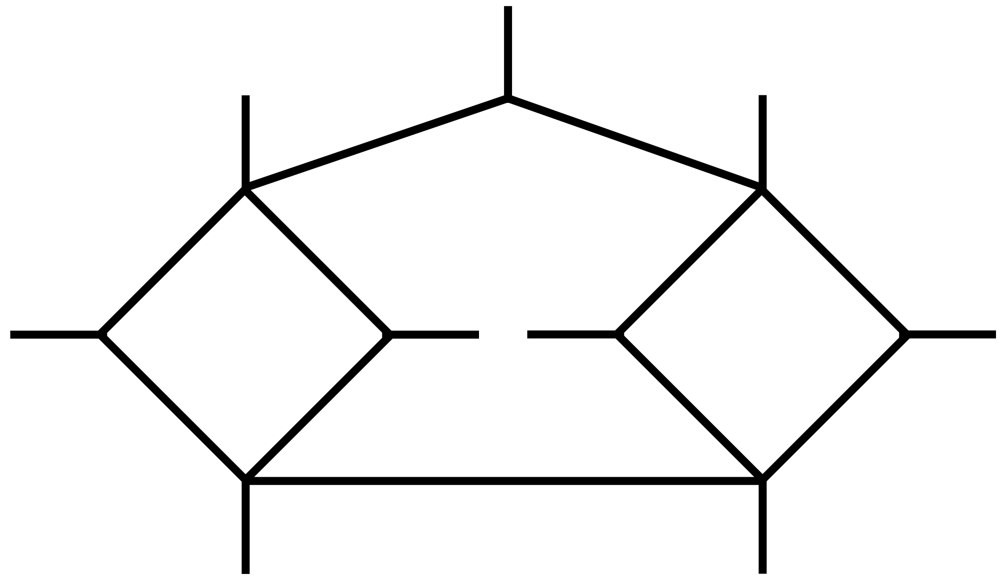
Feynman Integrals related to curves with genus >1 are known, but poorly studied

[Huang, Zhang, ’13]
[Hauenstein, Huang, Mehta, Zhang, ‘15]

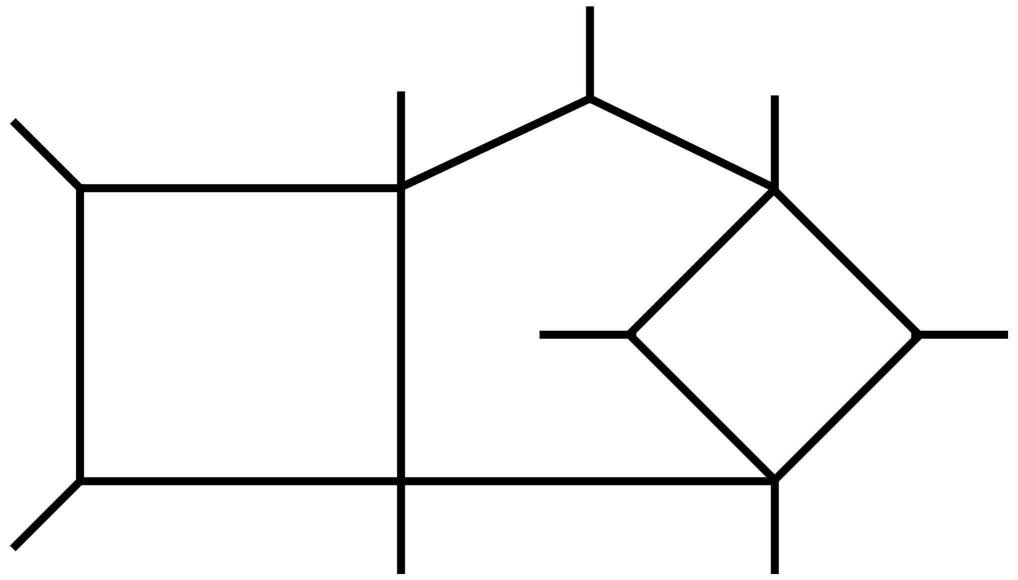
Genus 5



Genus 9

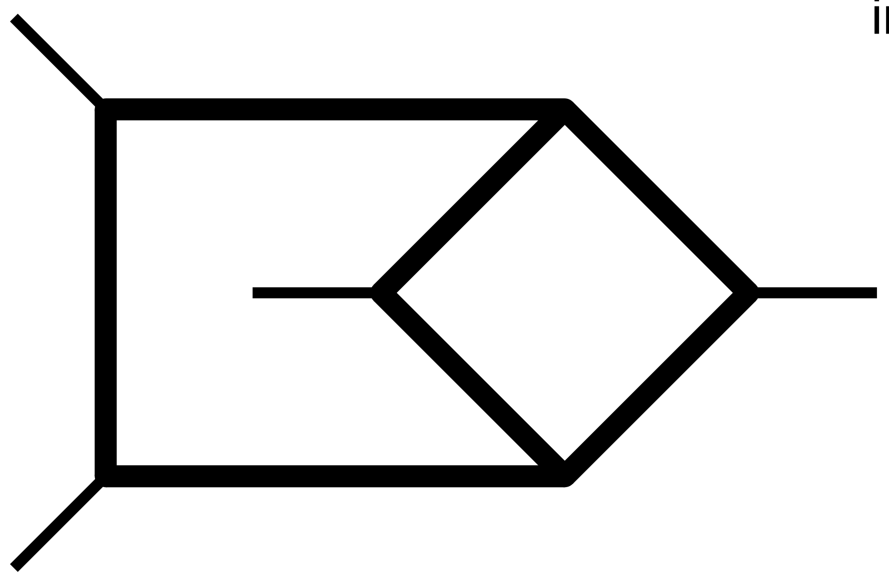


Genus 13



Simplest Example: Non-planar Crossed Box

in $D = 4$

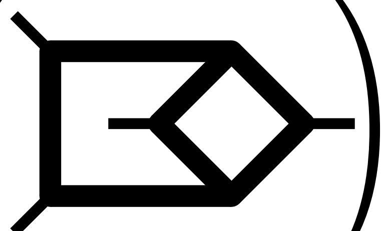


Genus 3

[Zhang, Georgoudis, ‘15]

Genus and Curve Representations

Extract curve via maximal cut

MaxCut in $D = 4$ $\Big($  $\Big)$

Momentum Representation
[Georgoudis, Zhang, 15']

$$\int \frac{\mathrm{d}z\; z}{\sqrt{P_8(z)}}$$

with

$$z = \mathrm{tr}_-(p_4 p_2 \ell_1 p_1)/s^2$$

Hyperelliptic curve of genus 3

Loop-by-loop Baikov Representation

$$\int \frac{\mathrm{d}z}{\sqrt{P_6(z)}}$$

with

$$z = (\ell_1 \cdot p_3)$$

Hyperelliptic curve of genus 2

Which one is it?
Why the discrepancy?

Curves with Automorphisms

Hyperelliptic curve of genus g $\mathcal{H} : y^2 = P_{2g+2}(z)$ **with automorphism group** $\mathrm{Aut}\,(\mathcal{H})$

Any $\mathcal{H}$ has **hyperelliptic involution** $e_0 : y \to -y$

Consider transformations $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C})$ $\quad z = \gamma[\hat{z}] \equiv \dfrac{a\hat{z}+b}{c\hat{z}+d}\,, \quad y = \hat{y}\dfrac{1}{(c\hat{z}+d)^{2g+2}}$

If there is a γ such that $\mathcal{H} : \hat{y}^2 = \hat{P}_{2g+2}(\hat{z}) = Q_{g+1}(\hat{z}^2) \equiv c(\hat{z}^2 - \hat{\alpha}_1^2)\dots(\hat{z}^2 - \hat{\alpha}_{g+1}^2)$

then $\mathrm{Aut}\,(\mathcal{H}_g)$ contains the **extra involution** $\mathrm{Aut}\,(\mathcal{H}) \ni e_1 : \hat{z} \to -\hat{z}$

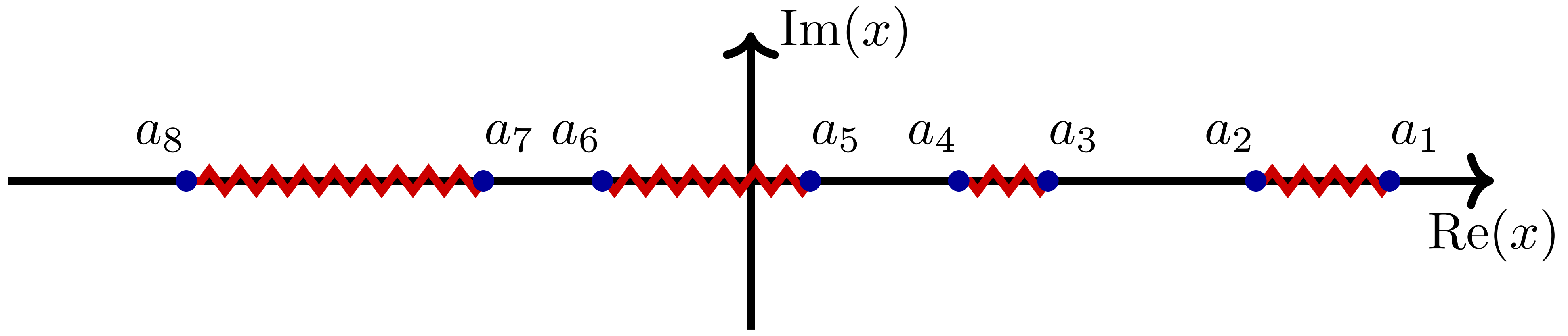
There are then two curves $\mathcal{H}_1$ and $\mathcal{H}_2$ with genus $\lfloor g/2 \rfloor$ and $\lceil g/2 \rceil$

$$\mathcal{H}_1 : v_1^2 = Q_{g+1}(w) = c(w - \hat{\alpha}_1^2)\dots(w - \hat{\alpha}_{g+1}^2)\,,$$
$$\mathcal{H}_2 : v_2^2 = wQ_{g+1}(w) = cw(w - \hat{\alpha}_1^2)\dots(w - \hat{\alpha}_{g+1}^2)$$

Recover $\mathcal{H}$ via $\begin{Bmatrix} \rho_1 : (v_1, w) \to (\hat{y}, \hat{z}^2) \\ \rho_2 : (v_2, w) \to (\hat{y}\hat{z}, \hat{z}^2) \end{Bmatrix}$ invariant under $\begin{cases} e_1 \\ e_0 \circ e_1 \end{cases}$

Example: Genus 3

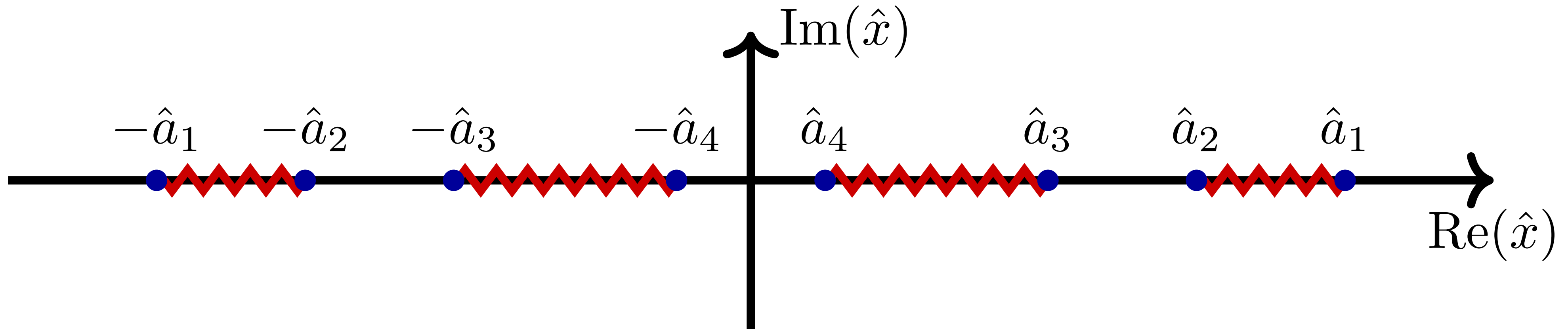
$$P_8(z) = 25z(z-2)(2z-1)(4z-3)(7z-4)(8z-1)(11z-2)(14z-3)$$



$$z = \gamma[\hat{z}]$$

$$\gamma = \begin{pmatrix} -1 & 1 \\ -3 & -2 \end{pmatrix}$$

$$\begin{aligned} \hat{P}_8(\hat{z}) &= (\hat{z}+1)(\hat{z}-1)(\hat{z}+2)(\hat{z}-2)(\hat{z}+3)(\hat{z}-3)(\hat{z}+4)(\hat{z}-4) \\ &= (\hat{z}^2-1)(\hat{z}^2-4)(\hat{z}^2-9)(\hat{z}^2-16) \end{aligned}$$



Genus 1 $\mathcal{H}_1$ $\quad v^2 = (w-1)(w-2^2)(w-3^2)(w-4^2)$

Genus 2 $\mathcal{H}_2$ $\quad v^2 = w(w-1)(w-2^2)(w-3^2)(w-4^2)$

Period Matrix

(Symplectic) Basis of contours $\Gamma_j \in (a_1, \ldots, a_g, b_1, \ldots, b_g)$

Basis of holomorphic differentials $\frac{z^{i<g}\mathrm{d}z}{\sqrt{P_{2g+2}(z)}}$

pairing

Period matrix

$$\mathcal{P}_{ij} = \int_{\Gamma_j} \frac{z^i \mathrm{d}z}{\sqrt{P_{2g+2}(z)}}$$

Extra involution leads to Relations between periods

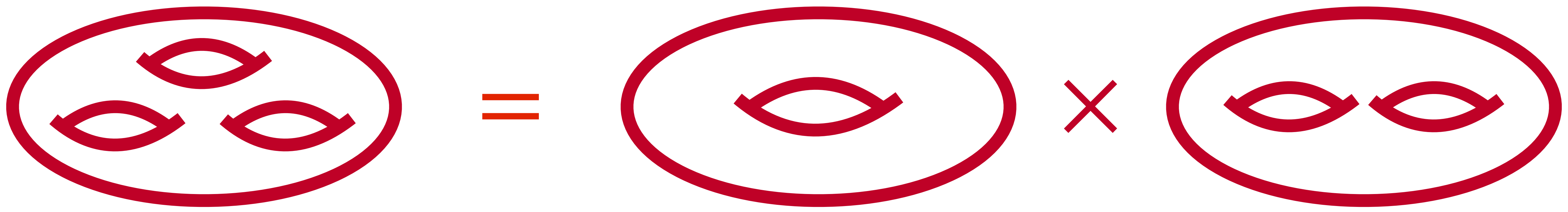
Can find $M_\Gamma \in \mathbb{Z}^{2g\times 2g}$, $M_\omega \in \mathbb{C}^{g\times g}$ such that

Period matrix of $\lfloor g/2 \rfloor$ curve

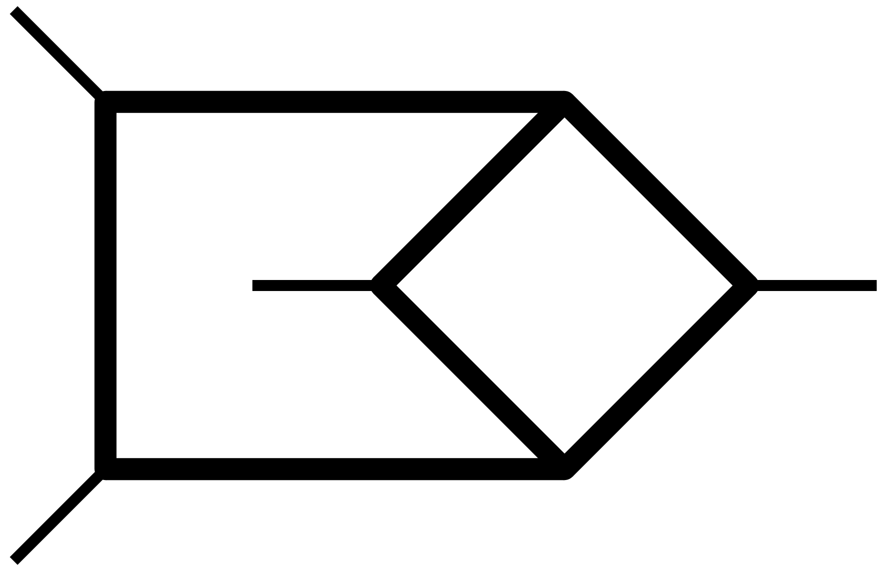
$$M_\omega^t \mathcal{P} M_\Gamma = \begin{pmatrix} \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_2 \end{pmatrix}$$

Period matrix of $\lceil g/2 \rceil$ curve

Period matrix becomes block diagonal



Feynman Integrals with Automorphisms



$r = \sqrt{s^3/(m^2 t(s+t))}$

$z = \frac{1}{r}\frac{\hat{z}-1}{\hat{z}+1},$

$P_8(z) \rightarrow Q_4(\hat{z}^2) = \hat{P}_8(\hat{z})$

Loop Momentum Representation

$w = \pm\sqrt{\hat{z}}$ → $Q_4(w)$

Genus 1

(generically $\lfloor g/2 \rfloor$)

$w = \pm\sqrt{\hat{z}}$, $\times w$ → $wQ_4(w) = Q_5(w) \xrightarrow{\simeq} P_6(z)$

Genus 2

(generically $\lceil g/2 \rceil$)

Baikov Representation

Maximal cut picks genus 2 curve

$$\int \frac{\mathrm{d}z \; z}{\sqrt{P_8(z)}} = r^2 \int \frac{\mathrm{d}w(w-1)}{\sqrt{wQ_4(w)}} = * \int \frac{\mathrm{d}z \; z}{\sqrt{P_6(z)}} + * \int \frac{\mathrm{d}z}{\sqrt{P_6(z)}}$$

Linear combination of genus 2 periods

Variable in [Georgoudis, Zhang] parametrisation:

$z = \mathrm{tr}_-(p_4 p_2 \ell_1 p_1)/s^2$

How does extra involution act on variable?

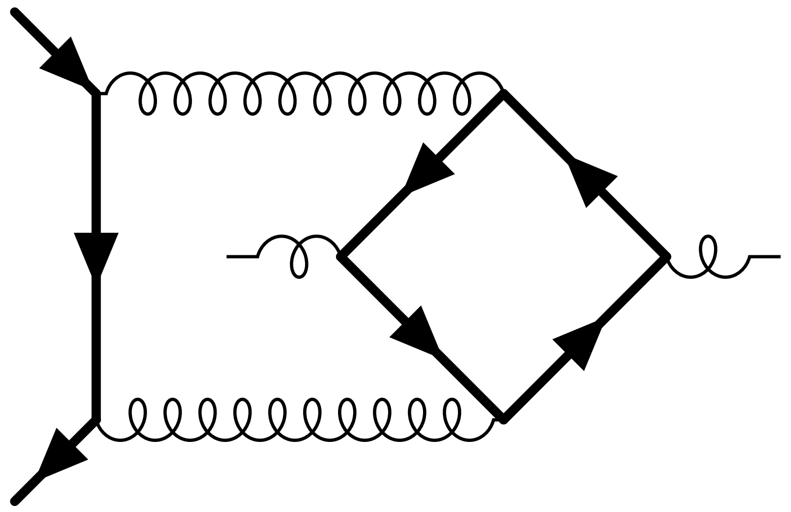
$$e_1\left[\mathrm{tr}_-(p_4 p_2 \ell_1 p_1)\right] = \mathrm{tr}_+(p_4 p_2 \ell_1 p_1)$$

Extra involution is discrete Lorentz transformation

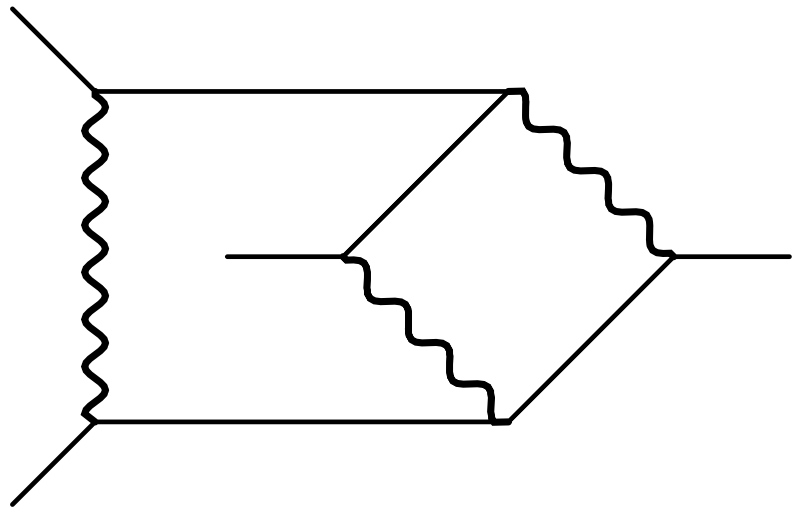
Choosing loop-momentum parametrisation invariant under (discrete) Lorentz transformations leads directly to genus 2 curve

Phenomenology and More Examples

1 Genus drop present in phenomenologically relevant integrals:

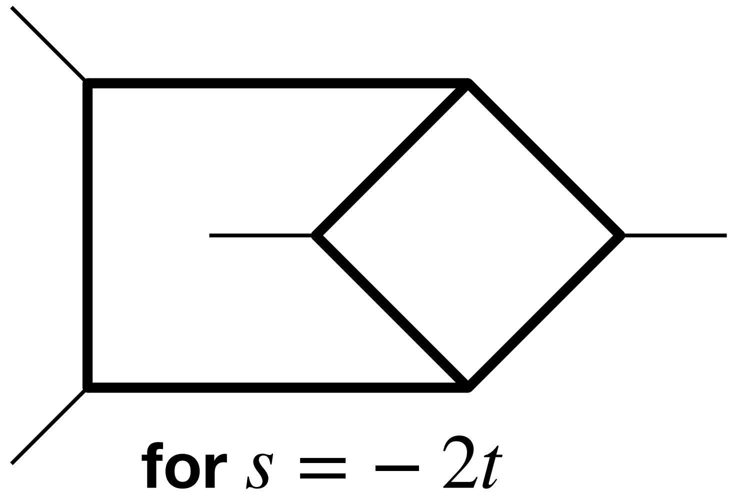


Already computed numerically
[Czakon, Mitov, 1303.6254]



Required for Møller scattering
$e^-e^- \to e^-e^-$ with three Z-

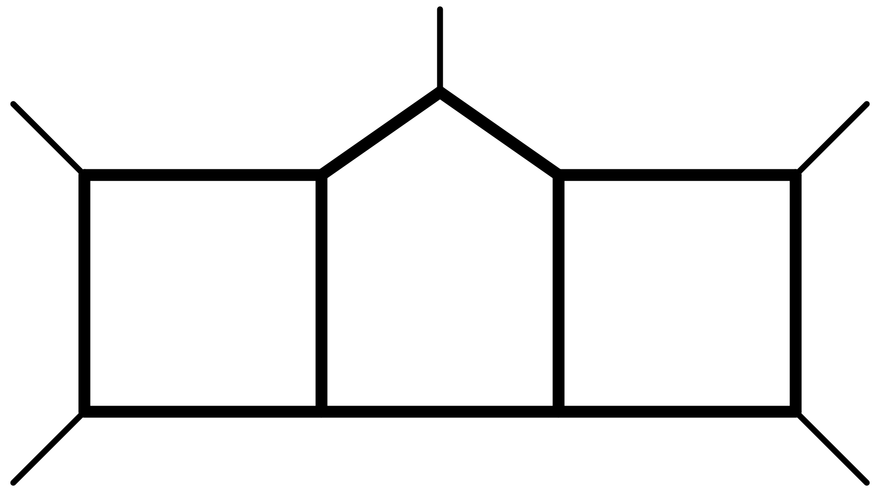
2 Genus drop in appears in kinematic limits:



for $s = -2t$

Genus 2 → Genus 1

3 Genus drop for non-hyperelliptic curve



Similar to integral in
[Huang, Zhang, 1302.1023]

Non-Hyperelliptic Genus 5 → Hyperelliptic Genus 3
(mechanism unclear yet)

Conclusions

- Beyond polylogs, control of geometry is crucial for evaluation of Feynman Integrals
- Integrals beyond elliptic ones are relevant to collider phenomenology today!
- Identification of “simplest” geometry not trivial (see genus reduction)
- There exists a wealth of mathematical knowledge for geometries associated that can be applied to Feynman integrals (algebraic curves studied since 19th century)

Backup

Calabi–Yau Operators

ℓ-loop Banana Integrals define special Calabi–Yau manifolds
Picard–Fuchs operator are called Calabi–Yau operators

Canonical coordinate:
q-coordinate or mirror-map

$$q = \exp(2\pi i \tau) \quad \tau = \frac{\omega_2}{\omega_1}$$

For $\ell = 2$
(*i.e.* sunrise/elliptic curve)
τ = ratio of periods
q = nome,

Picard–Fuchs operator in q-coordinate
Special Local Normal Form:
[M. Bogner, 13']

$$\mathcal{L}^{(2)} = \Theta_q^2$$

$$\mathcal{L}^{(3)} = \Theta_q^3$$

$$\mathcal{L}^{(4)} = \Theta_q^2 \frac{1}{Y_1} \Theta_q^2$$

$$\mathcal{L}^{(\ell)} = \Theta_q^2 \frac{1}{Y_1} \Theta_q \frac{1}{Y_2} \Theta_q \dots \Theta_q \frac{1}{Y_2} \Theta_q \frac{1}{Y_1} \Theta_q^2$$

Logarithmic derivative
$$\Theta_q = q \frac{\mathrm{d}}{\mathrm{d}q} = \frac{\mathrm{d}}{\mathrm{d}\log q} = \frac{1}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}\tau}$$

For $\ell = 4$:
Y_1 known as Yukawa coupling in string theory

Y_i: Y-invariants of operator

Calabi-Yau 3-fold from graph polynomial

$$F_{11111}^{(4)} = e^{4\varepsilon\gamma_E} \cdot \Gamma(1+4\varepsilon) \cdot \int\limits_{\alpha_i \geq 0} d^5\alpha\, \delta\left(1 - \sum_{i=1}^{5} \alpha_i\right) \frac{\mathcal{U}(\alpha)^{5\varepsilon}}{\mathcal{F}(\alpha)^{1+4\varepsilon}}$$

$$\mathcal{U}(\alpha) = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} + \frac{1}{\alpha_5}\right)$$

$$\mathcal{F}(\alpha) = x\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\mathcal{U}(\alpha)$$

$$\mathrm{CY}_3 = \left\{ [\alpha_1 : \alpha_2 : \alpha_3 : \alpha_4 : \alpha_5] \in \mathbb{CP}^4 \,\middle|\, x\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)\mathcal{U}(\alpha) = 0 \right\}$$

Frobenius Basis

$$
\begin{aligned}
\omega_1 &= \Sigma_1 \\
\omega_2 &= \log y\, \Sigma_1 + \Sigma_2 \\
\omega_3 &= \tfrac{1}{2} \log y^2\, \Sigma_1 + \log y\, \Sigma_2 + \Sigma_3 \\
\omega_4 &= \tfrac{1}{3!} \log y^3\, \Sigma_1 + \tfrac{1}{2} \log y\, \Sigma_2 + \log y \Sigma_3 + \Sigma_4
\end{aligned}
$$

$$\Sigma_i \in \mathbb{Q}[[y]]$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^2 from $A_{4,2}$:

$$L_3\, \omega = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^0 from $A_{4,4}$:

$$\frac{d \ln J}{dx} = \frac{d \ln \omega}{dx} + \frac{2\left(x^2 - 15x + 32\right)}{x\,(x-4)\,(x-16)}$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^{-1} from $A_{4,3}$ (plus previous):

$$\frac{1}{\omega}\frac{d^2\omega}{dx^2} - \frac{1}{2}\left(\frac{1}{\omega}\frac{d\omega}{dx}\right)^2 + \frac{2\left(x^2 - 15x + 32\right)}{x\left(x-4\right)\left(x-16\right)}\frac{1}{\omega}\frac{d\omega}{dx} + \frac{\left(x-8\right)}{2x\left(x-4\right)\left(x-16\right)} = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^{-1} from $A_{4,2}$:

$$\frac{d^2 F_{32}}{dx^2} + \left[\frac{d\ln\omega}{dx} + \frac{2\left(x^2 - 15x + 32\right)}{x\,(x-4)\,(x-16)}\right]\frac{dF_{32}}{dx} + \frac{3J}{2\pi i}\left[-\frac{(x-10)}{(x-4)\,(x-16)}\left(\frac{d\ln\omega}{dx}\right)^2 - \frac{2\left(x^3 - 30x^2 + 228x - 640\right)}{x\,(x-4)^2\,(x-16)^2}\frac{d\ln\omega}{dx} - \frac{\left(x^3 - 28x^2 + 168x - 384\right)}{x^2\,(x-4)^2\,(x-16)^2}\right] = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^0 from $A_{4,2}$:

$$\frac{dF_{42}}{dx} - 3F_{32}\frac{dF_{32}}{dx} + \frac{3J}{2\pi i}\frac{2(x-10)}{(x-4)(x-16)}\frac{dF_{32}}{dx}$$
$$+\frac{3J}{2\pi i}\left[\frac{2(x-10)}{(x-4)(x-16)}\frac{d\ln\omega}{dx} + \frac{2\left(x^3-30x^2+228x-640\right)}{x(x-4)^2(x-16)^2}\right]F_{32}$$
$$+\frac{J^2}{(2\pi i)^2}\left[-\frac{(11x+16)}{x^2(x-16)}\frac{d\ln\omega}{dx} - \frac{(11x-14)}{x^2(x-4)(x-16)}\right] = 0$$

Eliminating Non-Factorized Pieces

$$dI = \varepsilon \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,4} \\ A_{2,1} & A_{2,2} & A_{2,3} & A_{2,4} \\ A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\ A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4} \end{pmatrix} I$$

Remove ε^0 from $A_{4,3}$:

$$\frac{dF_{43}}{dx} + 2\frac{dF_{32}}{dx} + \frac{3J}{2\pi i}\left[-\frac{2\,(x-10)}{(x-4)\,(x-16)}\frac{d\ln\omega}{dx} - \frac{2\left(x^3-30x^2+228x-640\right)}{x\,(x-4)^2\,(x-16)^2}\right] = 0$$

Solution for Normalisation ω

First constraint is just Picard-Fuchs operator

$$L_3\,\omega = 0$$

Symmetric square

$$\omega_i \in \langle \psi_1^2,\, \psi_1\psi_2,\, \psi_2^2 \rangle$$

Second constraint is non-linear

$$\frac{1}{\omega}\frac{d^2\omega}{dx^2} - \frac{1}{2}\left(\frac{1}{\omega}\frac{d\omega}{dx}\right)^2 + \frac{2\left(x^2-15x+32\right)}{x\left(x-4\right)\left(x-16\right)}\frac{1}{\omega}\frac{d\omega}{dx} + \frac{(x-8)}{2x\left(x-4\right)\left(x-16\right)} = 0$$

$$\omega_i \in \langle \psi_1^2,\, \cancel{\psi_1\psi_2},\, \psi_2^2 \rangle$$

We choose: $\omega = \psi_1^2$

Next, Fix Kinematic Variable τ

With $\omega = \psi_1^2$ the constraint for τ is

$$\frac{d\ln J}{dx} = \frac{d\ln\omega}{dx} + \frac{2\left(x^2 - 15x + 32\right)}{x\left(x-4\right)\left(x-16\right)}$$

As hoped, satisfied by

$$\tau = \frac{\psi_2}{\psi_1} = \frac{\psi_2^{\text{sun}}}{\psi_1^{\text{sun}}} \qquad J = \frac{\psi_1^2}{W}$$

Wronskian $W = \psi_1 \frac{d}{dx}\psi_2 - \psi_2 \frac{d}{dx}\psi_1$

Constraints for F_{32}, F_{42}, F_{43}

Remaining differential equations are fulfilled for

$$F_{32} = F_2 - \frac{\pi i\,(x-10)}{(x-4)\,(x-16)\,W}\left(\frac{\psi_1}{\pi}\right)^2$$

$$F_{42} = \frac{3}{2}F_2^2 + \frac{\pi^2\,(x+8)^2\,(x^2-8x+64)}{8x^2\,(x-4)^2\,(x-16)^2\,W^2}\left(\frac{\psi_1}{\pi}\right)^4$$

$$F_{43} = -2F_2 - \frac{\pi i\,(x-10)}{(x-4)\,(x-16)\,W}\left(\frac{\psi_1}{\pi}\right)^2$$

All depend on one additional function F_2

F_2 has to satisfy

$$\frac{d^2F_2}{dx^2}+\left[\frac{2\left(x^2-15x+32\right)}{x\left(x-4\right)\left(x-16\right)}+2\left(\frac{d\ln\psi_1}{dx}\right)\right]\frac{dF_2}{dx}=\frac{\pi i\left(x-8\right)\left(x+8\right)^3}{x^2\left(x-4\right)^3\left(x-16\right)^3W}\left(\frac{\psi_1}{\pi}\right)^2$$

Solution: Iterated integral of meromorphic modular form of weight 6!

$$F_2=\left(2\pi i\right)^2\int_{i\infty}^{\tau}d\tau_1\int_{i\infty}^{\tau_1}d\tau_2\underbrace{\frac{x\left(x-8\right)\left(x+8\right)^3}{864\left(4-x\right)^{\frac{3}{2}}\left(16-x\right)^{\frac{3}{2}}}\left(\frac{\psi_1}{\pi}\right)^6}_{g_6}$$

Properties:

- $\bar{q}$ expansion of g_6 has only integer coefficients
- $\bar{q}^n$ coefficient of g_6 divisible by n^2
- Carrying out integration, F_2 has simple poles at $x=4,\ 16$

Basis of Modular Forms

Two classes

Holomorphic: $b_0 = \frac{\psi_1^{\text{sun}}}{\pi}$ $\qquad b_1 = y\frac{\psi_1^{\text{sun}}}{\pi}$

Meromorphic: $b_3 = \frac{1}{(y-3)}\frac{\psi_1^{\text{sun}}}{\pi}$ $\qquad b_{-3} = \frac{1}{(y+3)}\frac{\psi_1^{\text{sun}}}{\pi}$

Poles corresponding to (pseudo-)threshold

Can use these to express all modular forms appearing

Example:
$$f_{2,a} = \left(\frac{1}{x-4}+\frac{1}{x-16}\right)\frac{\psi_1^2}{2\pi i W}$$
$$= \left[\frac{1}{6}y^2 - \frac{5}{3}y + \frac{9}{2} - \frac{6}{y-3} - \frac{24}{y+3}\right]\left(\frac{\psi_1^{\text{sun}}}{\pi}\right)^2$$
$$= \frac{1}{6}b_1^2 - \frac{5}{3}b_0b_1 + \frac{9}{2}b_0^2 - 6b_0b_3 - 24b_0b_{-3}.$$

Letter $f_{2,b}$ is not a modular form, but iterated integral of one: non-trivial transformation under $\Gamma_1(6)$

Path decomposition gives us

$$
\begin{aligned}
(f_{2,b}|_2\gamma)(\tau) = f_{2,b}(\tau)& \\
&-6\frac{c}{c\tau+d}\frac{1}{2\pi i}I\left(1,1,g_6;\tau\right)+18\left(\frac{c}{c\tau+d}\right)^2\frac{1}{(2\pi i)^2}I\left(1,1,1,g_6;\tau\right) \\
&-24\left(\frac{c}{c\tau+d}\right)^3\frac{1}{(2\pi i)^3}I\left(1,1,1,1,g_6;\tau\right) \\
&+\frac{C_{1,6}}{(c\tau+d)^2}-\frac{2\pi i C_6}{c\left(c\tau+d\right)^3}
\end{aligned}
$$

Constants:

$$
\begin{aligned}
C_{1,6} &= I\left(1,g_6;i\infty,\frac{a}{c}\right) \\
C_6 &= I\left(g_6;i\infty,\frac{a}{c}\right)
\end{aligned}
$$

Singularities obstruct simple evaluation

E.g. $a/c = 1/6$: $C_{1,6}=5$, $C_6 = \frac{1620\zeta_3}{\pi^4} - i\frac{42}{\pi}$

Defining "Quasi-Eichler" of weight k, depth p:

$$(f|_k\gamma)(\tau) = f(\tau) + \sum_{j=1}^{p}\left(\frac{c}{c\tau+d}\right)^j f_j(\tau) + \frac{P_\gamma(\tau)}{(c\tau+d)^p}$$

$f_{2,b}$ transforms "Quasi-Eichler" of modular weight 2 and depth 3

Solution for Master Integrals

Initial condition of I_{1111} in limit $1/x \to 0$ from Mellin-Barnes representation

Master integrals to arbitrary power in ε as iterated integrals over $\{1, f_{2,a}, f_{2,b}, f_{4,a}, f_{4,b}, f_6\}$

$$I(f_1, ..., f_n; \tau) = (2\pi i)^n \int_{i\infty}^{\tau} d\tau_1 ... \int_{i\infty}^{\tau_{n-1}} d\tau_n \; f_1(\tau_1) ... f_n(\tau_n)$$

e.g., with $I_2 = \varepsilon^3 \frac{\pi^2}{\psi_1^2} I_{1111} = \varepsilon^3 I_2^{(3)} + \varepsilon^4 I_2^{(4)} + \mathcal{O}(\varepsilon^5)$

$$I_2^{(3)} = \frac{4}{3}\zeta_3 + I(1, 1, f_{4,a}; \tau)$$

Holomorphic, agrees with [Bloch, Kerr, Vanhove]

$$I_2^{(4)} = 2\zeta_4 + \frac{4}{3}\zeta_3 \left[\frac{11}{2}\ln(\bar{q}) - I(f_{2,a}; \tau) - I(f_{2,b}; \tau)\right] + \zeta_2 \ln^2(\bar{q}) - I(1, 1, f_{2,a}, f_{4,a}; \tau)$$
$$- I(1, f_{2,a}, 1, f_{4,a}; \tau) - I(f_{2,a}, 1, 1, f_{4,a}; \tau) - I(1, 1, f_{2,b}, f_{4,a}; \tau)$$
$$+ 2I(1, f_{2,b}, 1, f_{4,a}; \tau) - I(f_{2,b}, 1, 1, f_{4,a}; \tau)$$

Obtained explicit expressions for all master integrals up to ε^6

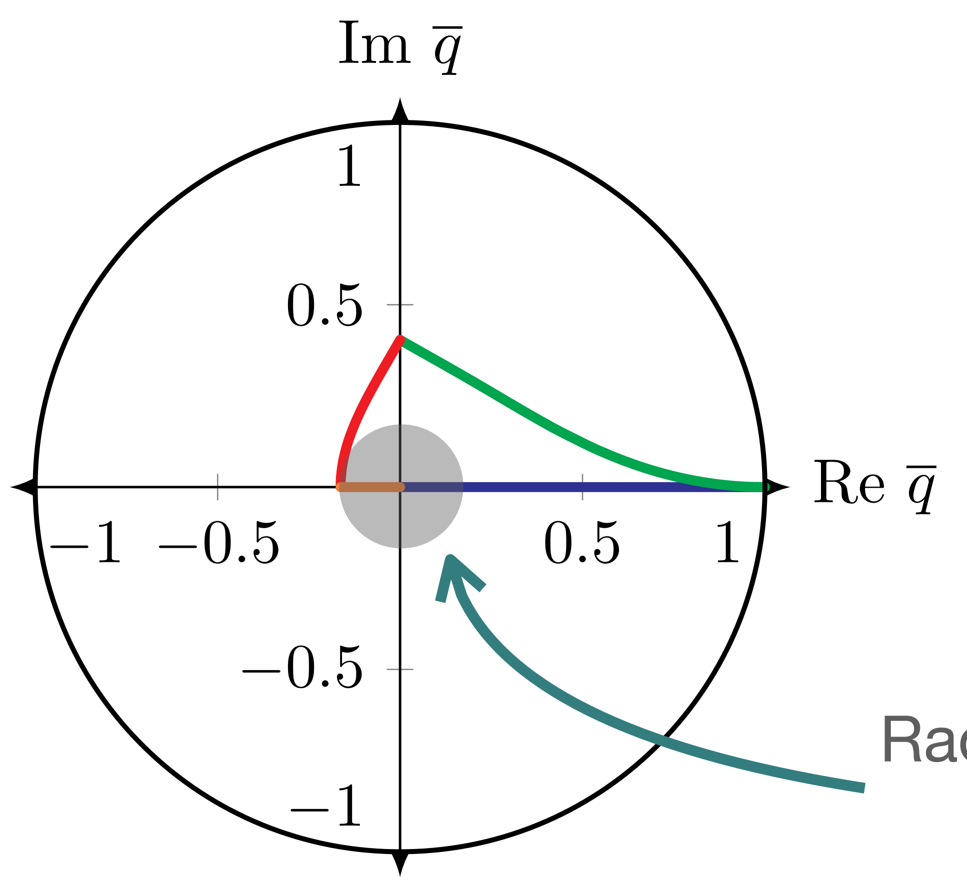
All integrals have uniform length

Numeric Verification

Numeric evaluation via $\overline{q}$-expansion

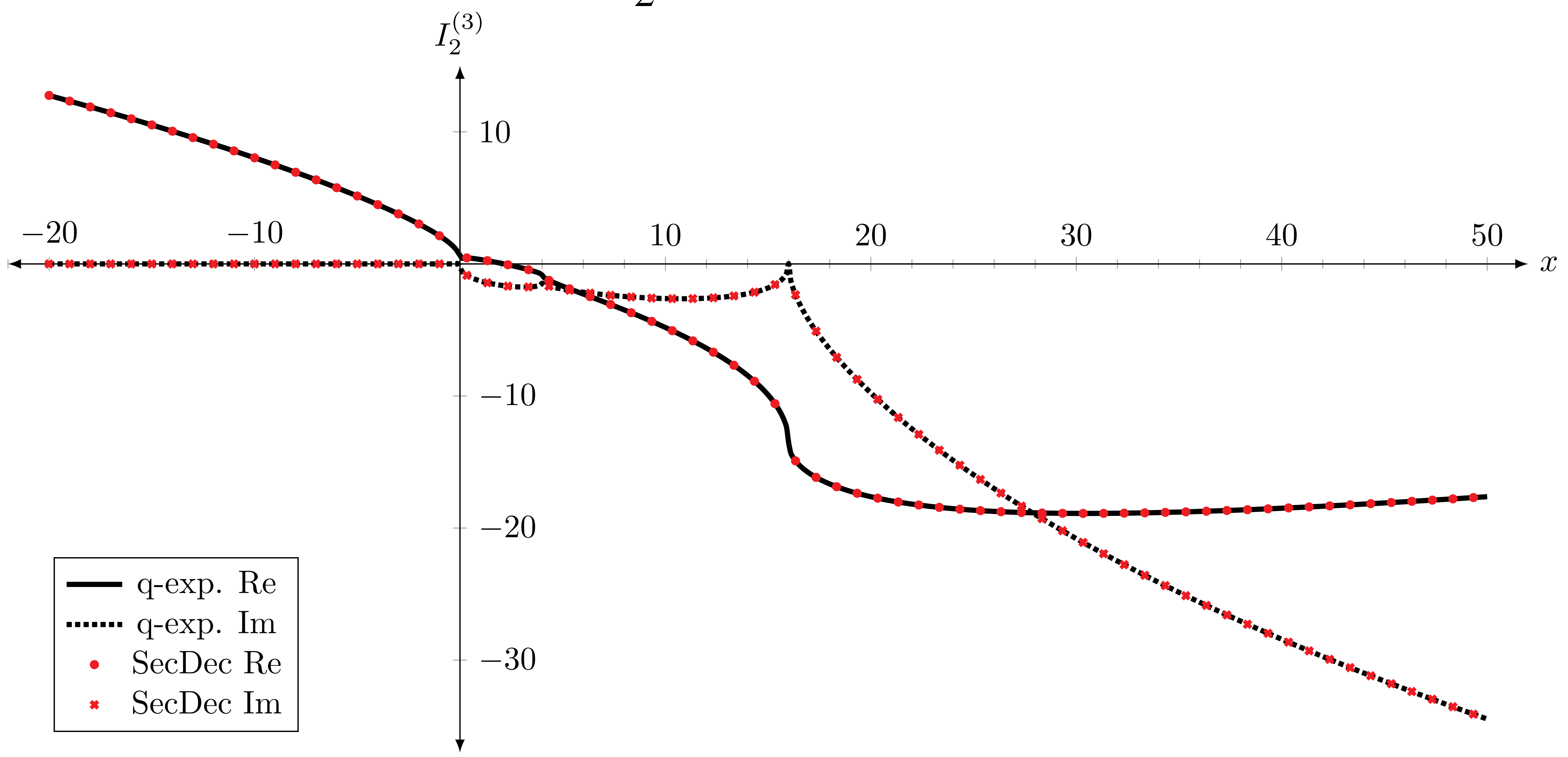
Singularities limit radius of convergence

Comparison against `SecDec`



Radius of convergence of $\overline{q}$ expansion

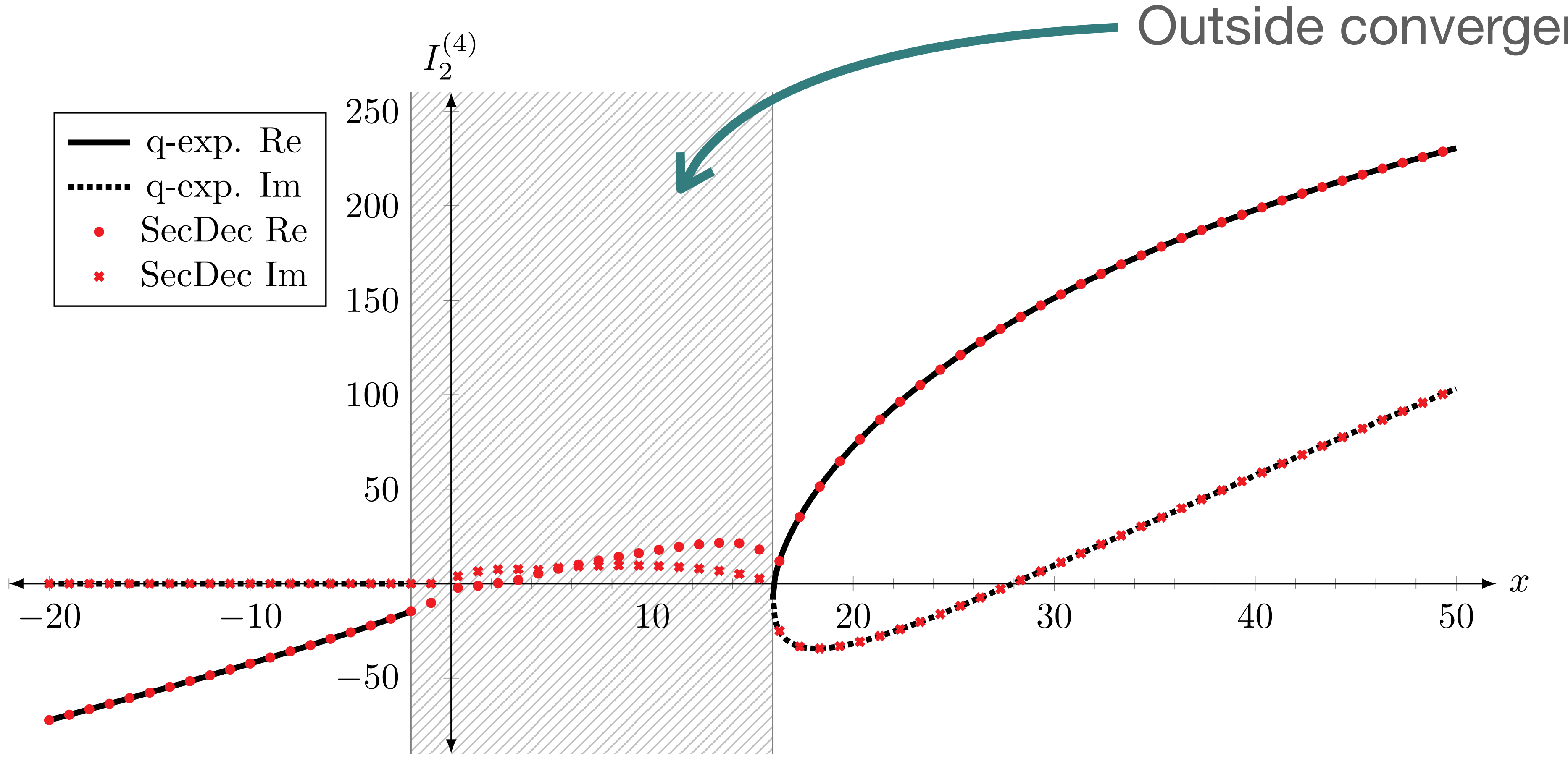
I_2, order ε^3



Only $f_{4,a}$, therefore holomorphic

I_2, order ε^4

Outside convergence radius



Also meromorphic $f_{2,a}, f_{2,b}$